SOME QUESTIONS OF INTERPOLATION BY MEANS OF ENTIRE FUNCTIONS

In the present paper we shall consider three interpolation problems in the class of entire functions.

1958

SovietRxiv

Abstract

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MATHEMATICS

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SOME QUESTIONS OF INTERPOLATION BY MEANS OF ENTIRE FUNCTIONS

(Presented by Academician M. V. Keldysh, 21 I 1958)

In the present paper we shall consider three interpolation problems in the class of entire functions.

Let $\{\lambda_n\}$, $0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_n| \leqslant \cdots$, $\lim_{n\to\infty} |\lambda_n| = \infty$, be a sequence of points in the plane of the complex variable z, and let $\{a_n\}$ be a sequence of complex numbers.

1. The class of entire functions having, for a given refined order of growth $\rho(r)$* ($\lim_{r\to\infty} \rho(r) = \rho < \infty$), normal or minimal type will be denoted by $[\rho(r), \infty)$.

The question is posed of the conditions that must be imposed on the sequence $\{\lambda_n\}$ of interpolation nodes in order that there should exist at least one function $f(z)$ of the class $[\rho(r), \infty)$ with the property

$$f(\lambda_n) = a_n, \tag{1}$$

provided only that the numbers $\{a_n\}$ $(n = 1, 2, \ldots)$ satisfy the condition

$$\overline{\lim_{n\to\infty}} \frac{\ln |a_n|}{|\lambda_n|^{\rho(|\lambda_n|)}} \leqslant \rho. \tag{2}$$

An analogous question was first posed and solved by A. F. Leont' ev ([1]) for entire functions of finite order and normal type.

For the formulation of the result we introduce the notion of the associated function of a sequence.

Let a sequence $\{\lambda_n\}$, $0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, $\lim_{n\to\infty} |\lambda_n| = \infty$, be given, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{n}{|\lambda_n|^{\rho(|\lambda_n|)}} < \infty, \tag{3}$$

where $\rho(r)$ is a refined order of growth.

For ρ nonintegral, the associated function for the given sequence $\{\lambda_n\}$ is called the canonical product

$$\Phi(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\frac{z}{\lambda_n}+\frac{z^2}{2\lambda_n^2}+\cdots+\frac{z^p}{p\lambda_n^p}\right)\quad([\rho]=p).$$

* A function $\rho(r)$ satisfying the conditions $\lim_{r\to\infty}\rho(r)=\rho$ and $\lim_{r\to\infty} r\rho'(r)\ln r=0$ is called a refined order of growth. If, for some entire function $f(z)$, the quantity

$$\sigma_f=\overline{\lim}\frac{\ln M_f(r)}{r^{\rho(r)}}\qquad\left(M_f(r)=\max_{|z|=r}|f(z)|\right)$$

is different from zero and infinity, then $\rho(r)$ is called a refined order of growth of the given function.

This same canonical product is called an associated function if ρ is an integer and the quantity

$$\delta=\overline{\lim_{r\to\infty}}\frac{1}{L(r)}\left|\frac{1}{\rho}\sum_{|\lambda_n|\le r}\frac{1}{\lambda_n^\rho}\right|\qquad\left(L(r)=r^{\rho(r)-\rho}\right)$$

is bounded.

If $\delta=\infty$, then as the associated function we shall take the function

$$\Phi(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\frac{z}{\lambda_n}+\frac{z^2}{2\lambda_n^2}+\cdots+\frac{z^\rho}{\rho\lambda_n^\rho}\right)\times$$

$$\times\left(1-\frac{z}{\mu_n}\right)\exp\left(\frac{z}{\mu_n}+\frac{z^2}{2\mu_n^2}+\cdots+\frac{z^\rho}{\rho\mu_n^\rho}\right),$$

where $\{\mu_n\}$ is a sequence of points of the z-plane satisfying the following four conditions:

$\alpha)$ the distance of any point of the sequence $\{\mu_n\}$ from the points $\left\{\lambda_n\exp\left(\frac{k\pi i}{\rho}\right)\right\}$ $(k=0,1,2,\dots,2\rho-1)$

is greater than $d|\lambda_n|^{1-\rho(|\lambda_n|)}$ for some d in the interval $(0,1)$;

$$\beta)\quad \lim_{n\to\infty}\frac{n}{|\mu_n|^{\rho(|\mu_n|)}}<\infty;$$

$$\gamma) \quad |\mu_{n+1}| - |\mu_n| > k|\mu_n|^{1-\rho(|\mu_n|)}, \quad \text{where } k > 0;$$

$$\delta) \quad \overline{\lim_{r\to\infty}} \frac{1}{L(r)} \left| \frac{1}{\rho} \left(\sum_{|\lambda_n|\le r} \frac{1}{\lambda_n^\rho} + \sum_{|\mu_n|\le r} \frac{1}{\mu_n^\rho} \right) \right| < \infty.$$

Lemma 1. *To every sequence $\{\lambda_n\}$ satisfying condition (3) there corresponds some associated function.*

The sequence $\{\mu_n\}$ satisfying the indicated conditions is, of course, not unique. To each sequence $\{\mu_n\}$ there will correspond its own associated function.

Theorem 1. *In order that, for every system of numbers $\{a_n\}$ satisfying condition (2), there exist at least one function $f(z) \in [\rho(r), \infty)$ with property (1), it is necessary that the sequence $\{\lambda_n\}$ satisfy conditions (3) and*

$$\overline{\lim_{n\to\infty}} \frac{1}{|\lambda_n|^{\rho(|\lambda_n|)}} \ln \left| \frac{1}{\Phi'(\lambda_n)} \right| < \infty, \tag{4}$$

*where $\Phi(z)$ is any associated function of the sequence $\{\lambda_n\}$, and it is sufficient that conditions (3) and (4) be fulfilled for at least one associated function.**

In view of the fact that, for a given sequence $\{\lambda_n\}$, the associated function is not unique, the following theorem is of interest:

Theorem 2. *Suppose there exists an entire function $\Phi_1(z)$ of the class $[\rho(r), \infty]$ such that*

$$\Phi_1(\lambda_n) = 0 \qquad (n = 1, 2, ...);$$

$$\overline{\lim_{n\to\infty}} \frac{1}{|\lambda_n|^{\rho(|\lambda_n|)}} \ln \left| \frac{1}{\Phi_1'(\lambda_n)} \right| < \infty.$$

* In the article by A. F. Leont' ev [1] an analogous theorem is proved for entire functions of finite order of normal type, but with the additional requirement of symmetry of the set of interpolation nodes. Here this requirement is absent. Recently it became known to us that A. F. Leont' ev, by another method, has also removed the requirement of symmetry of the interpolation nodes and expressed the necessary and sufficient conditions in other terms. This result of A. F. Leont' ev' s has not yet been published.

Then any adjoint function $\Phi(z)$ of the sequence $\{\lambda_n\}$ also satisfies the condition

$$\overline{\lim}_{n\to\infty}\frac{1}{|\lambda_n|^{\rho(|\lambda_n|)}}\ln\left|\frac{1}{\Phi'(\lambda_n)}\right|<\infty.$$

Thus, if there is at least one adjoint function of the sequence satisfying condition (4), then this condition is also satisfied by any other adjoint function of the given sequence.

2. Order and type are a rather crude characteristic of the growth of an entire function. A finer characteristic of growth is given by the indicator. We shall use the generalized indicator

$$h(\varphi)=\overline{\lim}_{r\to\infty}\frac{\ln|f(re^{i\varphi})|}{r^{\rho(r)}},$$

where $\rho(r)$ is the proximate order of growth of the given function.

Consider the interpolation problem in the class of entire functions with indicator not exceeding that of a given function. We have solved the problem for the case when the set of interpolation nodes $\mathfrak{M}$ is regularly distributed*. Among entire functions whose set of zeros coincides with the set $\mathfrak{M}$, which is regularly distributed for some exponent $\rho(r)$ ($\lim_{r\to\infty}\rho(r)=\rho$), the canonical functions are distinguished.

For nonintegral ρ, the canonical function is taken to be

$$F(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\frac{z}{\lambda_n}+\frac{z^2}{2\lambda_n^2}+\cdots+\frac{z^p}{p\lambda_n^p}\right),\qquad (p=[\rho]),$$

and for integral ρ, the function

$$F(z)=\exp(cz^\rho)\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\exp\left(\frac{z}{\lambda_n}+\frac{z^2}{2\lambda_n^2}+\cdots+\frac{z^\rho}{\rho\lambda_n^\rho}\right).$$

The growth indicator of the canonical function is called the indicator of the set $H(\theta)$ (see ([2]), Chap. II).

Let us pose the question of interpolating the sequence $\{a_n\}$ at the nodes $\{\lambda_n\}$ by entire functions with indicator not exceeding a given function $h(\theta)$. Obviously, such interpolation is possible in the case when

$$\overline{\lim}_{n\to\infty}\frac{\ln|a_n|}{|\lambda_n|^{\rho(|\lambda_n|)}h(\psi_n)}\leqslant 1\qquad(\psi_n=\arg\lambda_n).$$

It is impossible to choose $h(\theta)<H(\theta)$, since for the sequence of numbers $a_n=0$ the function $F(z)$ is a solution of the interpolation problem, and its indicator is

the least one for which there exists an entire function, not identically equal to zero, that vanishes on the regular set—

* The concept of a regularly distributed set of points was introduced by B. Ya. Levin (see ([2]), Chap. II, § 1). Let us recall the definition. A set $\mathfrak{M}$ of complex-plane points is called regularly distributed with exponent $\rho(r)$ ($\lim_{r\to\infty}\rho(r) = \rho$), for nonintegral ρ, if it has angular density for this exponent, i.e., if for all values φ and θ ($0 < \varphi < \theta \le 2\pi$), except possibly for a countable set, there exists the limit

$$\lim_{r\to\infty}\frac{n(r,\varphi,\theta)}{\rho(r)} < \infty,$$

where $n(r,\varphi,\theta)$ is the number of points of the set $\mathfrak{M}$ in the sector $|z| \le r,\ \varphi < \arg z < \theta$, and, for integral ρ, if besides the angular density of the set $\mathfrak{M}$ there exists a number c such that the quantity

$$\lambda(r) = \frac{1}{L(r)}\left\{c + \frac{1}{\rho}\sum_{|\lambda_n|\le r}\frac{1}{\lambda_n^\rho}\right\} \qquad (L(r) = r^{\rho(r)-\rho};\ \lambda_n \in \mathfrak{M})$$

has a limit as $r \to \infty$.

in the set $\{\lambda_n\}$ (see (1), Ch. IV, § 3). Thus, $H(\theta)$ is the smallest indicator for which the stated interpolation problem can be posed. We shall denote by M_H the class of entire functions with indicator not exceeding $H(\theta)$. It turns out that in the class of functions M_H the interpolation problem is solvable.

Theorem 3. Let the set of interpolation nodes $\{\lambda_n\}$ be regularly distributed with exponent $\rho(r)$ and with indicator of the set $H(\theta)$. In order that, for every system of numbers $\{a_n\}$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{|\lambda_n|^{\rho(|\lambda_n|)}H(\psi_n)} \le 1 \qquad (\psi_n = \arg\lambda_n),$$

there should exist at least one function $f(z) \in M_H$ with property (1), it is necessary and sufficient that the condition

$$\overline{\lim_{n\to\infty}}\frac{\ln\left|\dfrac{1}{F'(\lambda_n)}\right|}{|\lambda_n|^{\rho(|\lambda_n|)}H(\psi_n)} \le -1$$

be satisfied; here $F(z)$ is the canonical function.

3. The set of interpolation nodes $\{\lambda_n\}$ may be situated not in the whole plane, but inside some angle (α, β), and may have angular density with exponent $\rho(r)$ $(\lim_{r\to\infty} \rho(r) = \rho)$ inside this angle. In this case, when ρ is an integer, the problem arises of completing the given set to a regularly distributed one. We shall restrict ourselves to considering the problem for $\rho(r) = 1$. In this case, to each completion that generates a regularly distributed set there corresponds the canonical function of the finite system with a definite indicator diagram. We shall call that completion best to which there corresponds an indicator diagram of the least length.

Theorem 4. Every set $\{\lambda_n\}$ situated inside the angle (α, β) and having angular density with exponent $\rho(r) = 1$ inside this angle has a best completion to a regular one.

Let $F_1(z)$ be the canonical function of the best completion, and $H_1(\theta)$ its indicator. It is obvious that the solution of the interpolation problem at the nodes $\{\lambda_n\}$ lying inside the angle (α, β) should be sought in the class of entire functions with indicator not smaller than $H_1(\theta)$. We shall denote by N_H the class of entire functions with indicator not exceeding $H_1(\theta)$.

Theorem 5. Let the set of interpolation nodes $\{\lambda_n\}$ be situated inside the angle (α, β) and have angular density with exponent $\rho(r) = 1$ inside this angle. In order that, for every system of numbers $\{a_n\}$ satisfying the condition

$$\varlimsup_{n\to\infty} \frac{\ln|a_n|}{|\lambda_n| H_1(\psi_n)} \leq 1,$$

there should exist at least one function $f(z) \in N_H$ with property (1), it is necessary and sufficient that the condition

$$\varlimsup_{n\to\infty} \frac{\ln\left|\dfrac{1}{F_1'(\lambda_n)}\right|}{|\lambda_n| H_1(\psi_n)} \leq -1$$

be satisfied.

In conclusion I express my deep gratitude to B. Ya. Levin for posing the problem and for the assistance he rendered me in my work on the topic.

Received
30 III 1957

References Cited

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2. B. Ya. Levin, *Distribution of Zeros of Entire Functions*, 1956.