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Abstract

Full Text

MATHEMATICS

N. M. KOROBOV

NEW NUMBER-THEORETIC ESTIMATES

(Presented by Academician I. M. Vinogradov on 25 X 1957)

Let $f(x) = \alpha_1 x + \dots + \alpha_{n+1} x^{n+1}$, where α_ν $(\nu = 1, 2, \dots, n+1)$ are arbitrary real numbers; P and Q are integers; $P > 1$; $n > 1$. In the present note new estimates are indicated for trigonometric sums of the form

$$\sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)}. \tag{1}$$

and some applications of these estimates are given.

Theorem 1. If, for some fixed δ from the interval $0 < \delta < 0.5$, the condition $n + \delta \le r \le n + 1 - \delta$ is satisfied, where r is defined by the equality $P^r = |\alpha_{n+1}^{-1}|$, then there exist an absolute constant C and a constant $\alpha = \alpha(\delta)$ such that

$$\left| \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)} \right| < C P^{1 - \frac{\alpha}{(n \ln n)^{2.5}}}.$$

The proof of Theorem 1 is based on the use of an analogous theorem on the estimation of rational trigonometric sums, a brief announcement of which was given in my paper ([1]).

Trigonometric sums of the form (1), in which the length of the interval of summation satisfies the condition

$$|\alpha_{n+1}^{-1}|^{\frac{1}{n+1}} < P < |\alpha_{n+1}^{-1}|^{\frac{1}{n-1}}, \tag{2}$$

have various applications in analytic number theory.*
Let an estimate of the sum (1) have the form

$$\left| \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)} \right| < e^{c\theta_1(n)} P^{1 - \frac{\alpha}{\theta_2(n)}}, \tag{3}$$

where $\theta_1(n) \ge 1$; $\theta_2(n)$ is an increasing function of n; c and α are constants independent of n. It is easy to show that the more slowly, as $n \to \infty$, the

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function $\theta(n) = (\theta_1(n) + \ln n)\theta_2(n)$ increases, the more accurate are the results obtained in many questions by means of estimates of the form (3).

The best estimates (3) in the indicated sense were obtained by the method of I. M. Vinogradov:

$$\theta_1(n) = n\ln^2 n, \qquad \theta_2(n) = n^2 \ln n \quad \text{(I. M. Vinogradov (}^2\text{));}$$

$$\theta_1(n) = \ln n, \qquad \theta_2(n) = n^3 \ln n \quad \text{(Hua Loo-keng (}^3\text{)).}$$

* The length of the interval of summation in Theorem 1 obviously satisfies condition (2).

The results presented give close values for $\theta(n)$, satisfying, for any fixed $\varepsilon > 0$, the condition

$$n^3 < \theta(n) = o(n^{3+\varepsilon}). \tag{4}$$

Theorem 1 of the present paper leads, for $\theta(n)$, to the condition

$$n^{2.5} < \theta(n) = o(n^{2.5+\varepsilon}),$$

which makes it possible to improve all those number-theoretic estimates in which condition (4) was previously used in an essential way. Thus, for example, in ([1]) the estimates

$$\zeta(1+it) = O(\{\ln|t|\}^{5/7+\varepsilon}),$$

$$\pi(x) - \operatorname{li} x = O(xe^{-a\{\ln x\}^{7/12-\varepsilon}})$$

were indicated, and the corresponding refinement of the boundary of the real parts of the zeros of the zeta-function was given. The following assertions are also valid:

Theorem 2. *Let $\sigma(n)$ be the sum of the divisors of the number n. Then, for every fixed $\varepsilon > 0$,*

$$\sum_{n\le x} \sigma(n) = \frac{\pi^2}{12}x^2 + O(x\{\ln x\}^{5/7+\varepsilon}),$$

$$\sum_{n\le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x - \frac{1}{2}\ln x + O(\{\ln x\}^{5/7+\varepsilon}).$$

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Theorem 3. *Let $a_{ij} = a_{ji}$ be integers, and let $A(x)$ be the number of integer points lying inside the ellipsoid*

$$\sum_{i,j=1}^{4} a_{ij}x_i x_j \leq x. \tag{5}$$

Then, for every fixed $\varepsilon > 0$,

$$A(x) = \frac{\pi^2}{2\sqrt{D}}x^2 + O(x\{\ln x\}^{5/7+\varepsilon}),$$

where D is the determinant of the positive quadratic form (5).

Theorem 4. *Let $\varphi(n)$ be Euler' s function. Then, for every fixed $\varepsilon > 0$,*

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x\{\ln x\}^{5/7+\varepsilon}).$$

The results given above strengthen known estimates obtained in the works of N. G. Chudakov, A. Z. Val' fish, and in a number of other works (see ([3–8])). The proof of Theorems 2–4 basically coincides with the proof of analogous theorems in ([5,6]), with the replacement, in the appropriate places, of the estimates of trigonometric sums that lead for $\theta(n)$ to condition (4) by the estimates indicated in Theorem 1.

We note that Theorems 2–4 can without difficulty be somewhat refined by replacing in them $\{\ln x\}^{5/7+\varepsilon}$ by an expression of the form $\{\ln x\}^{5/7}\{\ln \ln x\}^{\gamma}$, with some $\gamma > 0$. Analogous refinements are also easily obtained in Theorems 2–4 of ([1]).

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Note: Figure translations are in progress. See original paper for figures.

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