ON THE UNSTEADY MOTION OF A GAS DISPLACED BY A PISTON, WITH ALLOWANCE FOR BACK PRESSURE

1958

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

N. N. KOCHINA and N. S. MELNIKOVA

ON THE UNSTEADY MOTION OF A GAS DISPLACED BY A PISTON, WITH ALLOWANCE FOR BACK PRESSURE

(Presented by Academician L. I. Sedov, 11 IV 1958)

The problem of a point explosion in a medium at rest is considered, taking into account the displacement of air by the products of the explosion. It is assumed that the motion of the gas masses is modeled by the expansion of a piston according to some law. The problem of the expansion of a piston with constant velocity was solved by L. I. Sedov (^{1,2}) and G. Taylor (^{3}). The problem of the expansion of a piston according to the power law

$$r_* = \frac{ct^{m+1}}{m+1}$$

for three values of m, without taking back pressure into account, was considered by N. L. Krasheninnikova (^{4}), and in 1955, for a wide range of numbers m, by the authors of the present paper.

We assume that the initial pressure p_1 is different from zero, and that the gas masses move like a piston according to the law

$$v_* = ct^m \left\{ 1 + \frac{(m-1)}{2(2m-1)} \left(\frac{\gamma p_1 \lambda_*}{\rho_1 c} \right)^2 At^{-2m} \right\}, \tag{1}$$

where c and m are constants; ρ_1 is the initial density; λ_* is the dimensionless radius of the piston ($\lambda_* = r_*/r_2$); A is a dimensionless constant determined from the calculation. The problem under consideration will not be self-similar, since among the determining parameters r, t, p_1, ρ_1, and c, three have independent dimensions.

Let us introduce the system of dimensionless variables:

$$f = v/v_2; \qquad R = \rho/\rho_2; \qquad P = p/p_2; \qquad \lambda = r/r_2;$$

$$q = \frac{\gamma p_1}{\rho_1} \frac{1}{(dr_2/dt)^2}; \qquad s = \ln \left[\left(\frac{\rho_1}{p_1} \right)^{\frac{m+1}{2m}} c^{\frac{1}{m}} r_2 \right]. \tag{2}$$

The index 2 refers to quantities behind the front of the shock wave.

The solution of the problem of a piston moving with some velocity reduces to finding the functions $f(\lambda, q)$, $R(\lambda, q)$, and $P(\lambda, q)$ in some region of the λ, q plane (where $0 \leq q \leq 1$). These functions satisfy the differential equations of one-dimensional unsteady gas motion (see (^{2}), p. 257), as well as the boundary and initial conditions:

$$f(1, q) = R(1, q) = P(1, q) = 1; \qquad f(\lambda, 0) = f_0(\lambda);$$

$$R(\lambda, 0) = R_0(\lambda); \qquad P(\lambda, 0) = P_0(\lambda), \tag{3}$$

where $f_0(\lambda)$, $R_0(\lambda)$, and $P_0(\lambda)$ are the corresponding functions for self-similar motion, found and studied in considering the piston problem without allowance for back pressure.

At the points of the piston, the function $f(\lambda, q)$ is equal to the ratio of the piston velocity v^* to the velocity of the particles behind the shock wave v_2.

Set

$$\begin{gathered} f(\lambda, q) = f_0(\lambda) + f_1(\lambda) q + \dots; \qquad R(\lambda, q) = R_0(\lambda) + R_1(\lambda) q + \dots; \\ P(\lambda, q) = P_0(\lambda) + P_1(\lambda) q + \dots; \qquad s = \frac{1}{\nu_1}(\ln A_0 q + A q) + \dots \end{gathered} \tag{4}$$

$$\left(\nu_1 = -\frac{2m}{m+1}; \qquad A_0 = (m+1)^{\frac{2m}{m+1}} \left|\gamma \lambda_*^{\frac{2}{m+1}}\right|\right),$$

where A and λ_* are constants entering equation (1).

Considering the linearized problem, i.e., neglecting in the equations and boundary conditions terms of order q^2 and higher, we obtain for the functions $f_1(\lambda)$, $R_1(\lambda)$, $P_1(\lambda)$ and the constant A the following system of linear differential equations $\left(a_1(\lambda) = \frac{2}{\gamma+1} f_0(\lambda) - \lambda\right)$:

$$\begin{gathered}
a_1 R_0 f_1' + \frac{\gamma - 1}{\gamma + 1} P_1' + \left(\frac{2 f_0'}{\gamma + 1} + \frac{\nu_1}{2} \right) R_0 f_1 + \left(a_1 f_0' - \frac{\nu_1 f_0}{2} \right) R_1 + \\
+ \frac{(\gamma^2 + 4\gamma - 1)}{2\gamma(\gamma + 1)} P_0' - \left[\nu_1 \left(1 + \frac{A}{2} \right) + \frac{2 f_0'}{\gamma + 1} \right] f_0 R_0 = 0; \\
- \frac{2}{\gamma + 1} R_0 f_1' + a_1 R_1' + \frac{2}{\gamma + 1} \left(\frac{\nu - 1}{\lambda} R_0 + R_0' \right) f_1 + \\
+ \left[\frac{2}{\gamma + 1} \left(\frac{\nu - 1}{\lambda} f_0 + f_0' \right) + \nu_1 \right] R_1 - \lambda R_0' - \frac{2\nu_1}{\gamma - 1} R_0 = 0; \\
a_1 (R_0 P_1' - \gamma P_0 R_1') + \frac{2}{\gamma + 1} (R_0 P_0' - \gamma P_0 R_0') f_1 - \gamma a_1 R_0' P_1 + \\
+ \left[a_1 P_0' - \nu_1 (\gamma + 1) P_0 \right] R_1 - \frac{2 f_0}{\gamma + 1} (R_0 P_0' - \gamma P_0 R_0') + \\
+ \nu_1 \left[\frac{4\gamma^2 - (\gamma - 1)^2}{2\gamma(\gamma - 1)} + A \right] P_0 R_0 = 0
\end{gathered} \tag{5}$$

(primes denote derivatives with respect to λ).

The boundary conditions at the shock wave, taking into account relations (4) and the conditions at the shock wave for the self-similar functions $f_0(1) = R_0(1) = P_0(1) = 1$, give

$$f_1(1) = R_1(1) = P_1(1) = 0. \tag{6}$$

Assuming that the piston velocity is determined by formula (1), by virtue of relations (2)–(4) we find the condition at the piston

$$f_1(\lambda_*) = f_0(\lambda_*). \tag{7}$$

Here λ_* denotes the dimensionless coordinate of the piston; in the linearized problem λ_* is equal to the dimensionless coordinate of the piston obtained in solving the self-similar piston problem (^{5,6}). The problem reduces to solving the system of differential equations (5) in the interval $\lambda_* < \lambda < 1$ with boundary conditions (6) and (7). The condition (7) at the piston can be satisfied by choosing the constant A, on which the solution depends. Since the constant A enters only the free terms of the equations, and moreover linearly, the solution of system (5) may be sought in the form

$$f_1 = f_{11} + A f_{12}; \qquad R_1 = R_{11} + A R_{12}; \qquad P_1 = P_{11} + A P_{12}. \tag{8}$$

Substituting (8) into (5), we obtain for the functions f_{1i}, R_{1i}, P_{1i} $(i = 1, 2)$ two systems of differential equations:

$$a_1 R_0 f'_{1i} + \frac{\gamma-1}{\gamma+1}P'_{1i} + \left(\frac{2f'_0}{\gamma+1}+\frac{\nu_1}{2}\right)R_0 f_{1i} + \left(a_1 f'_0 + \frac{\nu_1 f_0}{2}\right)R_{1i} + b_{1i} = 0;$$

$$-\frac{2R_0}{\gamma+1}f'_{1i} + a_1 R'_{1i} + \frac{2}{\gamma+1}\left(\frac{\nu-1}{\lambda}R_0 + R'_0\right)f_{1i} +$$

$$+\left[\frac{2}{\gamma+1}\left(\frac{\nu-1}{\lambda}f_0 + f'_0\right)+\nu_1\right]R_{1i} + b_{2i} = 0;$$

$$a_1(R_0 P'_{1i} - \gamma P_0 R'_{1i}) + \frac{2}{\gamma+1}(R_0 P'_0 - \gamma P_0 R'_0)f_{1i} -$$

$$-\gamma a_1 R'_0 P_{1i} + \left[a_1 P'_0 - \nu_1(\gamma+1)P_0\right]R_{1i} + b_{3i} = 0; \tag{9}$$

$$b_{11} = \frac{\gamma^2+4\gamma-1}{2\gamma(\gamma+1)}P'_0 - \left(\nu_1 + \frac{2f'_0}{\gamma+1}\right)f_0 R_0; \qquad b_{21} = -\lambda R'_0 - \frac{2\nu_1}{\gamma-1}R_0;$$

$$b_{31} = \frac{4\gamma^2-(\gamma-1)^2}{2\gamma(\gamma-1)}\nu_1 P_0 R_0 - \frac{2f_0}{\gamma+1}(R_0 P'_0 - \gamma P_0 R'_0);$$

$$b_{12} = \frac{\nu_1}{2}f_0 R_0; \qquad b_{22} = 0; \qquad b_{32} = \nu_1 P_0 R_0.$$

From the boundary conditions (6), taking into account that $A \neq 0$, we obtain the boundary conditions for the functions f_{1i}, R_{1i}, P_{1i}:

$$f_{1i}(1) = R_{1i}(1) = P_{1i}(1) = 0. \tag{10}$$

Let us note that the linearized equations of motion (5) have an integral analogous to that found by A. Sakurai (^{7}) and V. P. Korobeinikov (^{8}) in the problem of a strong explosion:

$$\frac{2\nu_1}{(\gamma+1)(\nu+\nu_1)}\frac{f_1}{a_1} + \left(\frac{\nu_1}{\nu+\nu_1}-\gamma\right)\frac{R_1}{R_0} + \frac{P_1}{P_0} =$$

$$= \frac{R_0^\gamma}{P_0}\int_1^\lambda \frac{P_0}{a_1 R_0^\gamma}\left\{\left(\frac{\nu_1}{\nu+\nu_1}-\gamma\right)\left(\lambda\frac{R'_0}{R_0}+\frac{2\nu_1}{\gamma-1}\right)+\nu_1\left(\frac{3\gamma-1}{2\gamma}-A\right)+\lambda\frac{P'_0}{P_0}\right\}d\lambda.$$

The solution of the system of first-order linear differential equations (9) with boundary conditions (10) was obtained by us for the cases $m = -0.4$ and $m =$

-0.5 ($\nu = 3,\ \gamma = 1.4$) by numerical integration. Near the piston ($\lambda = \lambda_*$), for the calculation one has to use asymptotic formulas. Introduce the notation:

$$u = 1 - \frac{\lambda_*}{\lambda}; \qquad a = \frac{\gamma}{2m + \nu\gamma(m+1)}; \qquad b = \frac{2m}{2m + \nu\gamma(m+1)};$$

$$r = \frac{2}{\gamma}[\gamma(b\gamma - 1) - 1]; \qquad k_1 = -\frac{2(\gamma+1)}{\gamma}; \qquad k_2 = \frac{2m(m+1)}{\gamma D \lambda_*};$$

$$k_3 = \frac{\nu(\gamma+1)^2(m+1)\lambda_*}{4\gamma(\gamma-1)aD}; \qquad k_{41} = \frac{(\gamma+1)[4\gamma + \nu(3\gamma-1)]\lambda_*}{4\gamma^2};$$

$$k_{42} = -\frac{\nu(\gamma+1)\lambda_*}{2\gamma}; \qquad m_1 = -\frac{4(\gamma+1)\alpha\lambda_* D}{m+1}; \qquad m_2 = 2b\gamma; \qquad (11)$$

$$m_3 = \frac{\nu(\gamma+1)^2\lambda_*^2}{4(\gamma-1)}; \qquad m_{41} = -\frac{(\gamma+1)(\gamma^2 + 4\gamma - 1)\lambda_*^2 D}{2\gamma(\gamma-1)(m+1)^2};$$

$$m_{42} = -\frac{\nu(\gamma+1)aD\lambda_*}{m+1}; \qquad n_1 = -\frac{2(\gamma-1)(m+1)a^{b+1}bC}{(\gamma+1)^2\lambda_*};$$

$$n_2 = \frac{2(\gamma-1)(m+1)a^{bbC}}{(\gamma+1)^2\lambda_*^2 D}; \qquad n_3 = [\nu(m+1) - 2m]a - 1;$$

$$n_{41} = \frac{(\gamma-1)(m+1)a^{b+1}bC}{\gamma+1}; \qquad n_{42} = 0;$$

C and D are certain constants known from the calculation of the self-similar problem.

From equations (9) we obtain asymptotic formulas for the functions $f_{1i}(u)$, $R_{1i}(u)$, and $P_{1i}(u)$:

$$f_{1i}(u) = C_{1i} + C_{2i} + (k_{4i} + C_{2i}r)u +$$

$$+\frac{[n_1\,(C_{1i} + C_{2i}) + n_{4i}]}{(\gamma b - 1)}\left\{-\frac{k_3 m_2}{b} + \frac{k_2 - k_3 m_2}{1-b}\right\}u^{1-b};$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$
\begin{aligned}
R_{1i}(u) = C_{3i} u^{-(\gamma+1)b} &+ \frac{\left[n_1\left(C_{1i}+C_{2i}\right)+n_{4i}\right]}{(\gamma b-1)} u^{-b-1} + \\
&+ \frac{1}{\gamma b}\left[\frac{n_3}{k_3}\left(-k_1 C_{1i}+m_3 C_{2i}\right)+n_1\left(k_{4i}+C_{2i} r\right)\right] u^{-b} + \\
&+ \frac{\left[n_1\left(C_{1i}+C_{2i}\right)+n_{4i}\right]}{(\gamma b-1) b(\gamma-1)}\left[-\frac{m_2\left(k_3 n_1+n_3\right)}{b}+\frac{n_1\left(k_2-k_3 m_2\right)}{1-b}\right] u^{-2b};
\end{aligned}
\tag{12}
$$

$$
P_{1i}(u) = \frac{-k_1 C_{1i}+m_3 C_{2i}}{k_3} + \left(m_{4i}+\frac{m_3}{k_3} C_{2i} r\right) u - \frac{m_2\left[n_1\left(C_{1i}+C_{2i}\right)+n_{4i}\right]}{b(\gamma b-1)} u^{-b};
$$

C_{1i}, C_{2i}, and C_{3i} are constants entering the asymptotic formulas (12).

Fig. 1

Fig. 2

After the functions $f_{1i}(\lambda)$, $R_{1i}(\lambda)$, and $P_{1i}(\lambda)$ have been found, the constant A is determined from equation (7) and the first of equations (8):

$$
A = \left[f_0(\lambda_*) - {}' f_{11}(\lambda_*)\right] / f_{12}(\lambda_*). \tag{13}
$$

In Figs. 1-3 the distribution of velocity, density, and pressure in the air behind the shock wave is presented. The solid curves correspond to the value $m = -0.5$, the dashed curves to the value $m = -0.4$. The curves corresponding to $q = 0$ pertain to the automobile problem.

We note that the authors have solved an analogous problem for the case $v_x = ct^m$, and also the problem of unsteady motions of water caused by the expansion of a piston with constant velocity.

In conclusion we express our gratitude to L. I. Sedov for a number of valuable suggestions.

Fig. 3

Received
24 III 1958

Fig. 3

Figure 3: Fig. 3

CITED LITERATURE

1. L. I. Sedov, *Prikl. matem. i mekh.*, 9, issue 4 (1945).
2. L. I. Sedov, *Methods of Similarity and Dimensionality in Mechanics*, Moscow, 1957.
3. G. I. Taylor, *Proc. Roy. Soc.*, A, 186, No. 100 (1946).
4. N. L. Krasheninnikova, *Izv. AN SSSR, OTN*, No. 8 (1955).
5. N. S. Burnova-Melnikova, *Investigation of the Problem of a Point Explosion*, Dissertation, Moscow, 1953.
6. V. P. Korobeinikov, N. S. Melnikova, *DAN*, 116, No. 2 (1957).
7. A. Sakurai, *J. Phys. Soc. Japan*, 8, No. 5 (1953).
8. V. P. Korobeinikov, *DAN*, 104, No. 4 (1955).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.