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Abstract

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MATHEMATICS

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EIGENVALUES AND EIGENFUNCTIONS OF CERTAIN FOURTH-ORDER DIFFERENTIAL OPERATORS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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Let two linear self-adjoint differential operators be given on the interval $[0, 1]$:

$$L_\eta = -\eta^2 \frac{d^2}{dx^2}\left(p_2(x)\frac{d^2}{dx^2}\right) + L_0, \qquad \text{where } L_0 \equiv -\frac{d}{dx}\left(p_1(x)\frac{d}{dx}\right),$$

where the coefficients $p_i(x)$ $(i = 1, 2)$ are strictly positive and i times continuously differentiable everywhere in $0 \le x \le 1$. We shall regard the parameter η as small.

In contrast to operators of this type considered in works $(^{1-3})$, here the signs of the coefficients at the highest derivatives in the full (L_η) and degenerate (L_0) operators are the same.

Consider the boundary-value problems

$$L_\eta(u) = \lambda u, \qquad u(0) = u(1) = 0, \qquad u^{(\sigma)}(0) = u^{(\sigma)}(1) = 0 \quad (\sigma = 1, 2); \quad \text{(I)}$$

$$L_0(y) = \lambda y, \qquad y(0) = y(1) = 0. \quad \text{(II)}$$

1. Problem (I) has zero as its eigenvalue for certain exceptional values of the parameter η, since the following theorem holds.

Theorem. The problem

$$L_\eta(u) = 0, \qquad u(0) = u(1) = 0, \qquad u^{(\sigma)}(0) = u^{(\sigma)}(1) = 0 \quad (\sigma = 1, 2)$$

has nontrivial solutions, i.e. there exist eigenvalues $\eta_{\sigma,n}$ of this problem, and they are equal to

$$\eta_{\sigma,n} = \frac{\nu}{(3-\sigma)n\pi} \qquad (\sigma = 1, 2;\; n = 1, 2, ...),$$

and, additionally, for the case $\sigma = 1$:

$$\eta_{1,n} = \eta_n^* \qquad (n = 1, 2, ...),$$

where

$$\nu = \int_0^1 \sqrt{\frac{p_1(x)}{p_2(x)}}\, dx,$$

η_n^* are the roots of the transcendental equation

$$\tan\frac{\nu}{2\eta} = \int_0^1 \frac{dt}{p_1(t)} \Big/ \eta \sum_{i=0}^{1} \sqrt{\frac{p_2(i)}{p_1^3(i)}}.$$

2. We introduce the stability region R of the solution of problem (I), defining it by the relations

$$\left|\sin\frac{\nu}{\eta}\right| \geq \frac{1 + (-1)^n \delta_{1\sigma}}{3-\sigma} M,$$

where $0 < M \leq 1$, n is the number of the point $\eta_n = \nu / n\pi$ $(n = 1, 2, ...)$, and additionally, for the case $\sigma = 1$,

$$\left|\operatorname{tg}\frac{\nu}{2\eta} - \int_0^1 \frac{dt}{p_1(t)} \Big/ \eta \sum_{i=0}^{1} \left|\sqrt{\frac{p_2(i)}{p_1^3(i)}}\right|\right| \geq N,$$

where $0 < N < +\infty$, i.e. the $\eta_{\sigma,n}$ and a certain neighborhood of the values of η around them are excluded from consideration. The stability region R is the set of intervals lying between two adjacent values $\eta_{\sigma,n}$, which, as $n \to \infty$ $(\eta \to 0)$, tend to the open interval $(\eta_{\sigma,n}; \eta_{\sigma,n+1})$, while at the same time tending to zero.

All further results are formulated only for $\eta \in R$.

3. The solution constructed in the stability region R of the problem

$$L_\eta(z) = f(x), \qquad z(0) = z(1) = 0, \qquad z^{(\sigma)}(0) = z^{(\sigma)}(1) = 0 \quad (\sigma = 1, 2),$$

where $f(x)$ is a function continuous together with its two derivatives in $0 \leq x \leq 1$, satisfying the boundary condition $f(0) = f(1) = 0$, has the form

$$z(x, \eta) = v(x) + \eta^{\sigma} \omega(x, \eta) + \zeta(x, \eta),$$

where $v(x)$ is the solution of the degenerate problem

$$L_0(v) = f(x), \qquad v(0) = v(1) = 0;$$

$$|\omega(x, \eta)| < O(1), \qquad |\omega'(x, \eta)| < \frac{1}{\eta} O(1),$$

and $\zeta(x, \eta)$ is a function of order $\eta^{\sigma+1}$, with first derivative of order η^{σ}.

4. Consider the problem of a more general form

$$L_{\eta}(J) = f(x), \qquad J(0) = J(1) = 0, \qquad J^{(\sigma)}(l, \eta) = S_{\sigma,l}(\eta)$$

$$(\sigma = 1, 2;\ l = 0, 1),$$

where $f(x)$ satisfies the same conditions as the right-hand side of the equation of the problem considered in the preceding item; $S_{\sigma,l}(\eta)$ is a constant satisfying the following condition: $|S_{\sigma,l}(\eta)| \leq C$, and may, in particular, be equal to zero. For its solution we obtain the estimate

$$|J(x, \eta)| < \max |f(x)| K_1(1 + \eta L_1) + \eta^{\sigma} S_{\sigma}(\eta) K_2(1 + \eta L_2),$$

$$|J'(x, \eta)| < \max |f(x)| K_3(1 + \eta L_3) + \eta^{\sigma-1} S_{\sigma}(\eta) K_4(1 + \eta L_4),$$

where

$$S_{\sigma}(\eta) = \max_{l} |S_{\sigma,l}(\eta)|;$$

K_i, L_i $(i = 1, 2, 3, 4)$ are certain constants.

This estimate is used to determine the order of smallness of the remainder term $\zeta(x, \eta)$ in the solution of the problem of the preceding item.

5. We construct the Green' s function $G(x, \xi, \eta)$ of the operator L_{η}. It turns out that, just as in work (1), the relations

$$G(x, \xi, \eta) = G_0(x, \xi) + \eta O(1) \quad \text{for } x, \xi \in [0,1],\ \eta \in R;$$

$$\frac{\partial G}{\partial x}(x, \xi, \eta) = O(1) \quad \text{for } x, \xi \in [0,1],\ \eta \in R,$$

hold, where $G_0(x, \xi)$ is the Green' s function corresponding to the degenerate differential operator L_0.

6. In the stability region R, for sufficiently small η, one can indicate the least positive eigenvalue λ of problem (I), which remains the least among the positive eigenvalues and does not

will merge with some other one upon a further decrease of η. We assign this eigenvalue the index 1. The corresponding normalized eigenfunction will be denoted by $u_1(x, \eta)$.

Similarly, for a sufficiently small value of the parameter $\eta \in R$, one can indicate a positive eigenvalue of problem (I), adjacent to $\lambda_1(\eta)$, which upon a subsequent decrease of η will remain the second (after $\lambda_1(\eta)$) positive eigenvalue and will not merge with any other eigenvalue of this problem, etc.

Thus, in the stability region R, for sufficiently small η, all positive eigenvalues of problem (I) and the corresponding eigenfunctions can be numbered, i.e.,

$$\lambda_1(\eta),\ \lambda_2(\eta), \dots, \lambda_k(\eta), \quad \text{where } \lambda_k(\eta) > 0;$$

$$u_1(x, \eta),\ u_2(x, \eta), \dots, u_k(x, \eta).$$

We note that, as η decreases, the number of positive eigenvalues of problem (I) increases.

The eigenvalues and normalized eigenfunctions of problem (II) will be denoted respectively by λ_k and $y_k(x)$.

7. In the stability region R, on the basis of the results obtained in the preceding items, using the variational method, we obtain

$$\lim_{\eta^{(m)} \to 0} \lambda_k(\eta^{(m)}) = \lambda_k, \qquad \lim_{\eta^{(m)} \to 0} u_k(x, \eta^{(m)}) = y_k(x) \quad (k = 1, 2, \dots),$$

where $\{\eta^{(m)}\}$ is any sequence of values of the parameter η, taken from the stability region R.

8. Using (with account taken of the stability region R) the method proposed in paper (1), we obtain representations for the positive eigenvalues and eigenfunctions of problem (I) in terms of the eigenvalues and eigenfunctions of the degenerate problem (II):

$$\lambda_k(\eta) = \lambda_k - \eta^{\sigma} \Phi_{\sigma,k} + \eta^{\sigma+1} O(1).$$

In particular, for $k = 1$ we have

$$\Phi_{\sigma,1} = \begin{cases} R(0)R(1)D_1^{-1}(\eta)\left\{\cos\frac{\nu}{\eta}\sum_{i=0}^{1}\sqrt{p_1(i)p_2(i)y_1^2(i)} + \varkappa\prod_{i=0}^{1}\sqrt[4]{p_1(i)p_2(i)y_1(i)}\right\}, & \text{for } \sigma = 1,\\ \int_0^1 p_2(\xi)y_1^2(\xi)\,d\xi, & \text{for } \sigma = 2, \end{cases}$$

where

$$D_1(\eta) = \sin\frac{\nu}{\eta}\int_0^1 \frac{dt}{p_1(t)} - \eta(h^2(0) + h^2(1))\left(1 - \cos\frac{\nu}{\eta}\right), \qquad R(i) = \sqrt{\frac{p_1(i)}{p_2(i)}},$$

$$h(i) = \sqrt[4]{\frac{p_2(i)}{p_1^3(i)}}\ (i = 0, 1), \qquad \varkappa = \frac{1}{R(0)R(1)}\int_0^1 \frac{dt}{p_1(t)}\left(\frac{1}{R(0)R(1)}\int_0^1 \frac{dt}{p_1(t)} - 1\right)\sin\frac{\nu}{\eta}{-2}.$$

$$u_k(x,\eta) = y_k(x) + \eta^{\sigma}\chi_{\sigma,k}(x,\eta) + \eta^{\sigma+1}O(1),$$

where $|\chi_{\sigma,k}(x,\eta)| < O(1)$.

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CITED LITERATURE

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