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Abstract

Full Text

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ON THE CONNECTION OF THE THEORY OF THE DISTRIBUTION OF ZEROS OF L-SERIES WITH THE ARITHMETIC OF TERNARY QUADRATIC FORMS

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The present note is a continuation of the note ([1]), in which a number of general theorems were formulated on the representation of large numbers m by positive ternary quadratic forms $f(x, y, z)$ of odd relatively prime invariants $[\Omega, \Delta]$. In the formulation of these theorems (with the exception of Theorems 1 and 4), in addition to the necessary genus conditions, there entered the condition

$$\left(\frac{-\Delta m}{q}\right) = 1, \tag{1}$$

where q is a prime number, introduced by the method of proof (and apparently not necessary). It turns out that this condition can be excluded from the formulations of Theorems 2, 3, and 5 of note ([1]), if one assumes the validity of the following hypothesis on the zeros of Dirichlet L-series (which is, evidently, an essential weakening of the extended Riemann hypothesis):

Hypothesis (H). For sufficiently large m, in the region

$$|s - 1| < \frac{(\ln \ln m)^2 \ln \ln \ln m}{\sqrt{\ln m}}$$

there are no zeros of Dirichlet L-functions of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\operatorname{Re} s > 1), \qquad \chi(n) = \left(\frac{-4\Omega^2 \Delta m}{n}\right), \tag{2}$$

where $\chi(n)$ is a character $(\operatorname{mod} 4\Omega^2 \Delta m)$.

Theorem. If hypothesis (H) is valid, then the conclusions of Theorems 2, 3, and 5 of note ([1]) remain in force if from their formulations one removes the requirement of the existence of a prime number q satisfying

$$\left(\frac{-\Delta m}{q}\right) = 1$$

(i.e. if one removes conditions (4), (7), and (12) of note ([1])).

We shall expand the formulation and give an outline of the proof of the assertion corresponding to Theorem 2 of note ([1]). Theorems 3 and 5 of ([1]) are modified and proved analogously.

Theorem 2a. Let hypothesis (H) be valid. Let $f(x, y, z)$ be an integral primitive positive ternary quadratic form of odd relatively prime invariants $[\Omega, \Delta]$; let m be an integer, relatively prime to $2\Omega\Delta$, for which the congruence $f(x, y, z) \equiv m \pmod{8\Omega\Delta}$ is solvable. Denote by $t(f, m)$ the number of primitive representations of the number m by the form f. Then there exist constants m_0, $\chi > 0$, and $\chi' > 0$, depending only on $\Omega\Delta$, such that for $m \geq m_0$

$$\chi h(-\Delta m) < t(f, m) < \chi' h(-\Delta m), \tag{3}$$

where $h(-\Delta m)$ is the number of classes of properly primitive positive binary quadratic forms of determinant Δm.

The path of proof of this theorem is close to the proof of the corresponding theorem ([1]). Here, too, the arithmetic of quaternions ([2]) is used. Only more precise estimates of divisor sums are required, carried out by the method of ([3]).

1°. The estimate (3) from above is trivial (see, for example, ([4])). Therefore we prove the estimate (3) only from below.

2°. From hypothesis (H), by arguments usual in the theory of L-functions, we derive that there exists a prime number q, not dividing $2\Omega\Delta m$, with the condition

$$\left(\frac{-\Delta m}{q}\right) = 1, \qquad q \leq \varkappa_1 \exp\left[\mu_1 \frac{\sqrt{\ln m}}{\ln \ln m}\right]. \tag{4}$$

The constants $\varkappa > 0$ and $\mu > 0$ here and below depend only on $\Omega\Delta$.

3°. Fix a sufficiently large natural number k such that there are $> \varkappa_2 n^2$ pairs of integral quaternions R_1 and R_2 of norm $r = q^k$ with the condition that $\overline{R_1} R_2$ is primitive. By Theorem 1 of note ([1]), there will be (for sufficiently large k) $> \varkappa_3 h(-\Delta m r^2)$ primitive vectors L of norm $\Delta m r^2$. Arguing analogously to ([5]), choose among them $n > \varkappa_4 h(-\Delta m r^2)$ inequivalent primitive vectors L_i of norm $\Delta m r^2$, for which the equalities

$$l + L_i = V_{iB} i \qquad (i = 1, 2, \dots, n), \tag{5}$$

hold, where B_i are integral primitive quaternions of norm r^s; V_i are integral quaternions of norm prime to r; $l = rl'$, where l' is an integer prime to r; for the integer s the inequalities

$$\varkappa_5 m^\rho \leq r^s < \varkappa_5 r m^\rho, \tag{6}$$

hold, where $0 < \rho \leq 1/2$ is a constant depending only on $\Omega\Delta$.

4°. Let w be the number of distinct B_i in the equalities (5), or in their part $i = 1, \dots, n'$ with the condition $n' > \varkappa_4' h(-\Delta m r^2)$. Then

$$w > \varkappa_6 m^\rho \exp\left[-\mu_2 \frac{\sqrt{\ln m}}{\ln\ln m}\right]. \tag{7}$$

This is the main part of the proof. The arguments are close in idea to ([5], [6]), but use more precise estimates of divisor sums by the method of ([3]).

5°. It turns out that among the equalities (5) there are $> n/2$ such equalities for which, for fixed i, there are $> \varkappa_7 s$ indices t with the condition

$$B_i = C_i^{(t)} R_1 R_2 A_i^{(t)}, \qquad N(A_i^{(t)}) = r^{4t}, \tag{8}$$

for otherwise the number w_1 of all distinct quaternions B of norm r^s is estimated from above as

$$w_1 < \varkappa_8 m^\rho \exp[-\mu_3 \sqrt{\ln m} \ln\ln m], \tag{9}$$

which, for sufficiently large m, contradicts the estimate (7).

6°. Among these $> n/2$ equalities (5) choose $> \varkappa_9 h(-\Delta m r^2)$ such equalities

$$rl' + L_i = V_{iB} i \qquad (i = 1, \dots, n_1 > \varkappa_9 h(-\Delta m r^2)); \tag{10}$$

that, for some fixed t, for all i

$$B_i = C_i^{(t)} R_1 R_2 A_i^{(t)}.$$

From the equalities (10) we immediately obtain the same number of equalities of the form

$$rl' + L_i' = R_2 D_{iR} 1, \qquad L_i' = (R_2 A_i^{(t)}) L_i (R_2 A_i^{(t)})^{-1} \qquad (i = 1, \dots, n_1); \tag{11}$$

L_i' are integral vectors of norm $\Delta m r^2$. Among them there will be

$$> \frac{x_9 h(-\Delta m r^2)}{x_{10} r^2} >$$

more than $x_{11} h(-\Delta m)$ distinct ones. Equality (11) shows that L_i' is divisible on the right by R_1, and on the left by R_2, and, since $R_1 R_2$ is primitive, L_i' is divisible by r, $L_i' = rL_i''$, where the L_i'' are already primitive vectors of norm Δm. But to these more than $x_{11}h(-\Delta m)$ distinct primitive vectors L_i'' of norm Δm there correspond, one-to-one, primitive representations of the number m by the form f.

Theorem 2a is proved.

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Note: Figure translations are in progress. See original paper for figures.