



Soviet-era science, translated into English

Mathematics

Academician P. Aleksandrov and Vl. Ponomarev

1958

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Abstract

Full Text

Mathematics

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On Bicomact Extensions of Topological Spaces

1. In the paper ⁽¹⁾, published in 1939 under the same title, the first of the authors of the present note gave a new method for constructing extensions of topological spaces and applied it, in particular, to a new construction of the maximal bicomact extension βX (the “Stone-Čech extension”) of a completely regular space X . Subsequently S. V. Fomin ⁽²⁾ applied the same method to the construction of a number of new extensions of topological spaces, thereby emphasizing its largely universal character. Essentially the same method is used also by Freudenthal in papers ^(3, 4), without, however, obtaining all bicomact extensions of the given space. The problem of constructing all bicomact extensions of a completely regular space was first solved completely by Yu. M. Smirnov ⁽⁵⁾, applying in a somewhat modified form the same method of paper ⁽¹⁾ and combining it with the notion of proximity in the sense of V. A. Efremovich. The method in question, and which we use throughout the present note, is based on the notion of “subordination,” or strengthened inclusion, $A < B$ of one set in another. In paper ⁽¹⁾, as applied to open sets, two kinds of subordination are considered: regular $H' < H$, if the closure $[H'] \subseteq H$, and completely regular $H' < H$, if $[H']$ is functionally separable from $X \setminus H$. After one or another subordination v has been introduced, a system of open sets $\xi = \{H\}$ is called proper in the sense of this subordination v , if for every $H \in \xi$ there is an $H' \in \xi$ subordinate to it. Centered proper systems $\xi = \{H\}$, maximal with respect to the conjunction of the two properties—centeredness and properness—are called “ends” in the sense of the subordination v , or simply v -ends. They are declared to be the points of the neighborhood space vX , whose open base consists of all sets O_H , where O_H , for a given open $H \subseteq X$, is defined as the set of all $\xi \in vX$ for which $H \in \xi$. It is natural to consider only those subordinations v for which the set of all neighborhoods O_x of any point $x \in X$ is a v -end. Then one may identify each point $x \in X$ with the end consisting of all neighborhoods in X of the point x , and this gives a topological inclusion of the space X into the space vX , so that vX becomes an extension of the space X . In the case of completely regular subordination v , it turns out that $vX \equiv \beta X$.
2. Freudenthal in paper ⁽³⁾ proposes, in essence, an axiomatic treatment of the notion of subordination for open sets. However, this path seems to us not wholly expedient: it is more convenient to axiomatize the subor-

dination of a closed set F to an open set H (which in fact is done for special cases also in ⁽¹⁾). We propose the following system of axioms for the subordination relation $F < H$ (where F always denotes a closed set, and H an open set in X):

K1 (the reflection axiom). If $F < H$, then $X \setminus H < X \setminus F$.

K2 (the first inclusion axiom). If $F < H$, then $F \subseteq H$.

K3 (second inclusion axiom). If $F \subseteq F_1 < H_1 \subseteq H$, then $F < H$.

K4 (addition axiom). If $F_1 < H_1$, $F_2 < H_2$, then $F_1 \cup F_2 < H_1 \cup H_2$.

K5 (refinement axiom). If $F < H$, then there exists an H_1 such that $F < H_1$, $[H_1] < H$.

K6 (trivial axiom). $\Lambda < \Lambda$ (where Λ is the empty set).

K7 (immersion axiom). For any neighborhood Ox of any point $x \in X$ there exists an O_1x such that $[O_1x] < Ox$.

It is easy to verify that a topological space in which one can define a subordination satisfying these axioms is completely regular, and that in every completely regular space one can define at least one subordination (for example, a completely regular one). Therefore, by a space we shall henceforth always mean a completely regular space.

3. We can now formulate a result which is essentially equivalent to the fundamental theorem of Yu. M. Smirnov ⁽⁵⁾:

Theorem 1. *For every subordination v given in a space X (and satisfying axioms K1–K7), the space vX is a bicomact extension of the space X . Conversely, every bicomact extension of the space X is a space vX for some subordination (satisfying axioms K1–K7); different subordinations yield different bicomact extensions.*

Let us outline a direct proof of Theorem 1. For any F , denote by Φ_F the set of all $\xi = \{H\} \in vX$ for which $H \cap F \neq \Lambda$ for every $H \in \xi$. Then $\Phi_F = X \setminus O_{X \setminus F}$ and $[O_H] = \Phi_{[H]}$. It is important to note (Lemma 1) that from $F < H$ it follows that $\Phi_F \subseteq O_H$. We now define in vX a subordination v^* , first setting $\Phi_F < O_H$ if $F < H$ in X , and then, for arbitrary closed Φ and open Γ in vX , setting $\Phi < \Gamma$ if there exists $F < H$ in X such that $\Phi \subseteq \Phi_F$, $O_H \subseteq \Gamma$. It is easy to verify that axioms K1–K7 are satisfied for the subordination v^* . Let us now call a neighborhood OF in X refining if there exists an O_1F such that $[O_1F] < OF$ in X . From the axioms of subordination it follows easily that F is the intersection of all its refining neighborhoods.

The bicomactness of the regular (by virtue of the first lemma) space vX is proved as follows. Let $\sigma = \{\Phi\}$ be a centered system of closed sets in vX . Then

$$\bigcap_{\Phi \in \sigma} \Phi = \bigcap_{\Phi \in \sigma} \bigcap_{\alpha} O_{\alpha} \Phi,$$

where $O_\alpha\Phi$ ranges over all refining neighborhoods of the set Φ . Intersecting all $O_\alpha\Phi$ with X , we obtain a centered proper system $\{H\}$, which, extended to a maximal ξ , gives (Lemma 2) a point ξ contained in all $O_\alpha\Phi$, $\Phi \in \sigma$, and hence also in their intersection $\bigcap_{\Phi \in \sigma} \Phi$, which is therefore nonempty.

This proves the bicomactness of vX .

It is easily proved that different subordinations v_1 and v_2 correspond to different spaces v_1X and v_2X , and that in every bicomactification there is a unique—so-called elementary—subordination, coinciding with the inclusion. If bX is an arbitrary bicomact extension of the space X , then the elementary subordination in bX determines a subordination in X by the rule: $F < H$ if $[F]_{bX}$ is contained in the (largest) open set Γ in bX cutting H out of X . For this subordination $vX \equiv bX$.

4. The question arises of a possible reduction of the stock of sets F, H such that the subordination $F' < H'$, defined for sets of this stock, extends to all of X (i.e. to arbitrary F, H) by the rule: $F < H$ if $F \subseteq F' < H' \subseteq H$.

As a first special case, consider all possible canonical closed and open sets F' and H' in X . In this case all the axioms are the same except K4, which is replaced by its weakening:

K\$' §4. If $F'_1 < H'_1$, $F'_2 < H'_2$, then $F'_2 \cup F'_1 < I([H'_1 \cup H'_2])$, where I denotes the open kernel.

If one considers the subordination “in the system of canonical sets” and extends it to the whole space X , then one obtains a subordination satisfying axioms K1-K7. Conversely, if a subordination is given for arbitrary F, H , then, by considering it only on canonical sets, one obtains a subordination of canonical sets; extending this, one returns to the original subordination.

5. The second special case is the case of some complete base (S. V. Fomin⁽²⁾, N. A. Shanin⁽⁶⁾) of the space X , i.e. a base \mathfrak{B} such that from $H \in \mathfrak{B}$ it follows that $X \setminus [H] \in \mathfrak{B}$, and from $H_1 \in \mathfrak{B}$, $H_2 \in \mathfrak{B}$ it follows that $H_1 \cup H_2 \in \mathfrak{B}$.

Suppose that a subordination v , satisfying axioms K1-K7, is defined for all sets H belonging to a complete base and for their closures $[H]$. Extending this subordination to arbitrary F, H , we obtain a subordination satisfying axioms K1-K7. *If the complete base \mathfrak{B} is a large base (i.e. for any F, OF there is an $H \in \mathfrak{B}$ such that $F \subseteq H \subseteq [H] \subseteq OF$), then any subordination in X can be obtained by extending some subordination defined in \mathfrak{B} .*

6. Let now \mathfrak{B} be a complete **peripherally bicomact base** in X (this means that $\text{fr } H = [H] \setminus H$, for every $H \in \mathfrak{B}$, is bicomact). E. Sklyarenko⁽⁷⁾ proved (correcting a result of Freudenthal⁽³⁾, valid only partially) a proposition which we may formulate as follows:

Theorem of E. Sklyarenko. Let $\mathfrak{B} = \{u_\alpha\}$ be a complete, peripherally bicomact base. Extending to the whole X the elementary subordination $[u_\beta] \subseteq$

u_α , defined in the base \mathfrak{B} (i.e. putting $F < H$ if there exists $u_\alpha \in \mathfrak{B}$ such that $F \subseteq u_\alpha \subseteq [u_\alpha] \subseteq H$), we obtain a subordination v in X , for which the bicomact vX has a zero-dimensional remainder $N = vX \setminus X$ in the sense that $\text{Ind } N = 0$.

We shall now supplement this result. Call a set N **zero-dimensionally lying** in a bicomact B if there exists a base $\mathfrak{B} = \{\Gamma\}$ of this bicomact such that $N \cap \text{fr } \Gamma = \Lambda$ for all $\Gamma \in \mathfrak{B}$. In fact, Sklyarenko' s paper contains a proof that, under the hypotheses of his theorem, the remainder $N = vX \setminus X$ is not only zero-dimensional, but also zero-dimensionally lies in vX . On the other hand, from Freudenthal' s paper ⁽³⁾ one can extract a quite correct proof that, if for a bicomact extension bX the remainder $bX \setminus X$ zero-dimensionally lies in X , then X is peripherally bicomact, i.e. has a peripherally bicomact base. Finally, let bX be a bicomact extension with remainder N zero-dimensionally lying in bX . Take a base of the space bX whose elements' boundaries have empty intersection with N . This base (which we may assume complete) cuts out from X a peripherally bicomact base \mathfrak{B} , defining in it the elementary subordination and extending it to all of X ; we obtain as the space vX precisely the given bX . Thus the following strengthening of Sklyarenko' s theorem holds:

Theorem 2. *In order that the space X have a bicomact extension with a remainder zero-dimensionally lying in it, it is necessary and sufficient that X be peripherally bicomact. Then every such bicomact extension can be obtained by extending to the whole X an elementary subordination defined in some complete peripherally bicomact base of the space X .*

7. A space X is zero-dimensional in the sense that $\text{Ind } X = 0$, if it has a large base consisting of open-and-closed sets. It is natural to call a set N **strongly zero-dimensionally lying** in a bicomact B if in B there exists a base $\mathfrak{B}^* = \{\Gamma\}$ such that $N \cap \text{fr}_B \Gamma = \Lambda$ for all $\Gamma \in \mathfrak{B}^*$, and the sets $N \cap \Gamma$ form a large base in N . Obviously, if a set strongly zero-dimensionally lies in some bicomact, then $\text{Ind } N = 0$. A partial converse to this is:

Theorem 3. *If $\text{Ind } X = 0$, then X strongly zero-dimensionally lies in each of its bicomact extensions.*

Indeed, if $\text{Ind } X = 0$, then in it there is a large base $\mathfrak{B} = \{H\}$, consisting of open-and-closed sets, and this base may be assumed complete. An arbitrary bicomact extension vX may be obtained on the basis of an ordering defined only in \mathfrak{B} . The corresponding sets O_H form a base in vX that cuts out from X the base \mathfrak{B} , and

$$X \cap \text{fr}_{vX} O_H = \text{fr}_X(X \cap O_H) = \text{fr}_X H = \Lambda,$$

as was required to prove.

Thus, the property of a space being strongly zero-dimensional in some bicomact extension of it is topologically invariant and equivalent to its zero-dimensionality

in the sense that $\text{Ind } X = 0$.

Department of Higher Geometry and Topology
of Moscow State University
named after M. V. Lomonosov

Received
6 V 1958

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Note: Figure translations are in progress. See original paper for figures.

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