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Abstract

Full Text

Mathematics

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ON THE APPROXIMATE CONSTRUCTION OF CERTAIN QUASICONFORMAL MAPPINGS

(Presented by Academician M. A. Lavrent' ev, 15 X 1957)

We consider the problem of constructing a quasiconformal mapping of the disk $T,\ |z|<1$, onto a domain G in the w-plane by means of a function

$$w = w(z) = u(x,y) + iv(x,y), \tag{1}$$

satisfying the linear elliptic system

$$\begin{aligned} au_x + bu_y &= v_y, \\ -bu_x + au_y &= -v_x, \end{aligned} \qquad a(x,y) > 0, \tag{2}$$

under the normalization conditions

$$w(0) = w_0 = 0, \qquad w(t_1) = w_1 = u_1 + iv_1, \tag{3}$$

where $w_0 \in G,\ w_1 \in \Gamma,\ |t_1| = 1.$ The curve Γ is the boundary of the domain G and is given by the equation

$$F(u,v) = 0. \tag{4}$$

It is assumed that a, b, and F are twice continuously differentiable functions of their arguments (the first two in the disk $\overline{T}$), and that the second derivatives of F in a neighborhood of the line Γ satisfy the Lipschitz condition

$$|F''_{u^p v^q}(u_1,v_1) - F''_{u^p v^q}(u_2,v_2)| \leq$$

$$\leq K\left(|u_1-u_2| + |v_1-v_2|\right), \qquad p+q=2, \quad p,q=0,1,2; \tag{5}$$

$$|\operatorname{grad} F| \geq \lambda > 0. \tag{6}$$

Let, on the boundary, $w(t) = u(t) + iv(t)$, $t = e^{i\varphi}$. Introduce in the space H_μ (of functions continuous in the sense of Hölder, with the usual norm; see (^{1})) the linear functionals k and l:

$$k[u(t)] = u(0,0), \qquad l[v(t)] = v(0,0) \tag{7}$$

and the operator S of passage to the conjugate function, analogous to the integral operator with the Hilbert kernel in the conformal case:

$$v(t) = Su(t), \qquad l[Su(t)] \equiv 0. \tag{8}$$

The actual computation of k and S requires solving the Dirichlet problem for the elliptic equation of second order in u that follows from system (2). The functional l is computed analogously, and it is evident that $\|k\| =$

$\equiv \|z\| \equiv 1$. For a general investigation of the operator S, the results of I. N. Vekua ([2], problem A) are applicable.

The mapping problem under consideration reduces to determining the functions $\{u(t), v(t)\}$, $t = e^{i\varphi}$, from the system of equations

$$F[u(t), v(t)] = 0, \tag{9}$$

$$L \equiv v(t) - Su(t) = 0, \qquad K[u(t)] = 0, \tag{10}$$

$$L_1 \equiv f_1 u(t_1) + g_1 v(t_1) + h_1 = 0 \tag{11}$$

(here in (11) $L_1(u,v) = 0$ is the exact or approximate equation of the normal to the curve Γ at the point w_1); in what follows

$$f_1 = -F_v[u_0(t_1), v_0(t_1)], \qquad g_1 = F_u[u_0(t_1), v_0(t_1)], \qquad h_1 = -f_1 u_1 - g_1 v_1. \tag{12}$$

Moreover, the solution must satisfy the following conditions: a) as φ varies by 2π, the point $w = u(t) + iv(t)$ must traverse the contour Γ once in the positive direction; b) since the normal (11) intersects the line Γ, besides w_1, at least at one more point (we denote by w_2 the intersection point nearest to w_1), one more condition is needed which would single out from the solutions of the system (9)—(11) the one satisfying (3), for example: $|w(t_1) - w_1| < |w_2 - w_1|$. Conditions a) and b) are taken into account in choosing the initial approximation (see below).

To solve the system (9)—(11) we apply Newton' s method, developed for the approximate solution of nonlinear functional equations by L. V. Kantorovich

([3-4]). Without loss of generality, we shall assume that the initial approximations $u_0(t)$ and $v_0(t)$ satisfy equations (10); the general case reduces to this by replacing $u_{0,1} = u_0 - K[u_0]$, $v_{0,1} = v_0 - L[u_0, v_0]$.

For definiteness, let us write the equations corresponding to the modified Newton method. Suppose that the n-th approximation $\{u_n(t), v_n(t)\}$, $n = 0, 1, 2, \ldots$, has been found. To determine the corrections

$$u_{n+1} - u_n = u', \qquad v_{n+1} - v_n = v' \tag{13}$$

we arrive at the linear boundary-value problem for the system (2) (instead of u', v' we again write u, v):

$$F_u[u_0(t), v_0(t)]u(t) + F'_v[u_0(t), v_0(t)]v(t) = -F[u_n(t), v_n(t)] \tag{14}$$

with the additional conditions

$$u(z) + iv(z)\big|_{z=0} = 0, \tag{15}$$

$$f_1 u(t_1) + g_1 v(t_1) = -L_1[u_n(t_1), v_n(t_1)]. \tag{16}$$

Since the system (2), by means of the "affine" substitution $u_1 = au,\ v_1 = v - bu$, is reduced to the Carleman system, it becomes possible, for the investigation of the boundary-value problem (14), to apply the theory developed by I. N. Vekua for the linear boundary-value problem A ([2]).

Lemma 1. The index of problem A corresponding to problem (14) for system (2) is equal to $n = 1$.

As is known, in this case the general solution contains $2n + 1 = 3$ arbitrary real constants, which here are determined by means of the three conditions (15)–(16).

Lemma 2. The homogeneous problem corresponding to (14), with condition (15), for $n = 1$ has one and only one independent solution, nowhere, except for $z = 0$, vanishing.

The proof follows from the proof of one proposition of the work ([2]) (p. 283) concerning the case $n = 0$. We denote the considered

in the lemma the solution by $W_0(z)$, normalizing it by $|W_0(t_1)| = 1$, and by $W_1(z)$ and $W_2(z)$—some solutions of the homogeneous problem corresponding to (14), without condition (15), satisfying the conditions: $W_1(0) = 1, \quad W_2(0) = i$. The existence of these two solutions follows from Lemmas 1 and 2.

Lemma 3. *Problem* (14) *with conditions* (15)–(16) *has one and only one solution for any right-hand sides.*

One of the possible ways of actually determining the corrections is connected with solving, by the method of successive approximations, the integral equation (8.96) of paper (^{2}), which here is conveniently written in the form

$$[I - L_{(A_0)}]V(z) = \Phi(z),$$

where $L_{(A_0)}$ is an integral operator depending on a certain function A_0. From the results of (^{2}) it follows that there exists a bounded inverse operator

$$[I - L_{(A_0)}]^{-1},$$

mapping the space H_μ (of functions Hölder-continuous in the circle $\overline{T}$, with the usual norm) into itself.

Introduce the notation:

$$A(t) = \frac{1}{2a}\frac{\partial}{\partial \overline{z}}(a - ib);$$

$$\Omega(z) = -\frac{1}{\pi}\iint_T \frac{A(t)\,dT}{t - z} = \rho(z) + i\omega_1(z);$$

$$\alpha = \frac{1}{a}\left[F_u(u_0, v_0) + bF_v(u_0, v_0)\right], \qquad \beta = F_v(u_0, v_0);$$

$$[(\alpha(t) - i\beta(t))e^{\Omega(t)}]_{|t|=1} = t^{-1}M(t)e^{i\omega_2(t)};$$

$$p_k(z) = \frac{1}{2\pi i}\int_{|t|=1} \omega_k(t)\frac{t + z}{t - z}\frac{dt}{t} = \omega_k^*(z) + i\psi_k^*(z);$$

$$A_{0,k}(z) = \overline{A}(z)e^{2i[\omega_k^*(z) - \omega_k(z)]}, \qquad k = 1, 2;$$

$$\left\|[I - L_{(A_{0,k})}]^{-1}\right\| \leq B_k;$$

$$c(t) = -\frac{e^{-\psi_2^*(t)}}{M(t)}, \qquad \gamma_n(t) = c(t)F[u_n(t), v_n(t)];$$

$$N_0 = \max\left[\frac{1}{\mu\pi}(1 + 2^{1+\mu} + 3^\mu) + \frac{\pi}{1 - \mu}; \frac{4^{1+\mu}}{\mu\pi}\right] + \frac{2\pi^{\mu-1}}{\mu};$$

$$B_0 = 4(1 + \|W_0^F(z)\|)(1 + \|W_1(z)\| + \|W_2(z)\|)\|c(t)\| \times$$
$$\times \left(1 + \left\|\frac{1 + ib(z)}{a(z)}\right\|\right) \left\|e^{\Omega(z) - ip_2(z)}\right\| B_1 N_0 + \frac{1}{\lambda}\|W_0(z)\|;$$

$$N_1 = 4e^{4\|\rho(t) + \psi_1^*(t)\|}\|a(t)\|B_2 N_0 + 2\|b(t)\|;$$

$$\eta_0 = \max\{\|F[u_0(t), v_0(t)]\|,\ |L_1[u_0(t_1), v_0(t_1)]|\};$$

$$K_0 = K(\|u_0(t)\| + \|v_0(t)\| + 2r)(1 + N_1)^2 + \max|F''_{u^p v^q}|,$$

$$p + q = 2; \qquad p, q = 0, 1, 2;$$

$$r = 2B_0\eta_0\sqrt{1 + N_1^2}.$$

We formulate the conditions for convergence of the approximations, following from Theorem 1 of paper (⁵).

Let the initial approximation correspond to a closed curve without multiple points

$$\Gamma_0: \quad w = u_0(t) + iv_0(t).$$

Construct an annular neighbor-

the part R of the line Γ_0 formed by the circle of radius r, whose center runs along Γ_0, and suppose that in R conditions (5)–(6) are satisfied.

Theorem. Suppose the inequality

$$h_0 = B_0^2 K \bar{\gamma}_0 < \frac{1}{2}.$$

is satisfied. Then in the domain

$$\|u(t) - u_0(t)\| \leq 2B_0\bar{\gamma}_0, \qquad \|v(t) - v_0(t)\| \leq 2N_1 B_0 \bar{\gamma}_0$$

there exists a solution $\{u^*(t), v^*(t)\}$ of the system of equations (9)–(11), and the successive approximations of the modified Newton process converge to this solution, with

$$\|u_n(t) - u^*(t)\| \leq \frac{q^n}{1-q} B_0 \bar{\gamma}_0, \qquad q = 1 - \sqrt{1 - 2h_0} < 1.$$

For conditions a) and b) to hold it is sufficient that

$$r < \min\left\{\min_{|t|=1} |u_0(t) + iv_0(t)|;\ |w_2 - w_1| - |w_0(t_1) - w_1|\right\}.$$

In particular, the indicated method is applicable to the approximate construction of conformal mappings. In this case, under the usual normalization conditions $w(0) = 0,\ w'(0) > 0$, the process of successive approximations resulting from the solution of the Riemann–Hilbert boundary-value problem can be expressed explicitly as

$$u_{n+1}(t) + iv_{n+1}(t) = u_n(t) + iv_n(t) +$$

$$+ te^{-ip_2(t)}\{\gamma_n(t) + iS\gamma_n(t) - i(k[\gamma_n(t)])\operatorname{tg} k[\omega_2(t)]\} \tag{17}$$

(in the present case $a \equiv 1,\ b = \Omega = 0$, S is the singular integral operator with Hilbert kernel; by Gauss' s theorem the functional k reduces to averaging; here the condition $\cos k[\omega_2(t)] \neq 0$ is necessary, similar to that which appears when Newton' s method is applied to the nonlinear singular equation of Theodorsen' s method; this condition is fulfilled for a sufficiently good initial approximation).

Another application of Newton' s method to the approximate construction of conformal mappings in connection with the method of conjugate trigonometric series was recently proposed by G. A. Nikolaeva ([6]).

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Note: Figure translations are in progress. See original paper for figures.

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