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G. M. IDLIS

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Abstract

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G. M. IDLIS

A GENERAL EXPRESSION FOR THE PHASE DENSITY OF FINITE STATIONARY AXISYMMETRIC SELF-GRAVITATING STELLAR SYSTEMS AND THEIR DIFFERENTIAL AXIAL ROTATION

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The phase density Ψ, being (by Liouville' s theorem $D\Psi/Dt = 0$) an integral of motion, may be represented in the form of an arbitrary function of any complete system of corresponding independent integrals of motion (Jeans' theorem (^{1})). In self-gravitating stellar systems with a stationary axisymmetric potential

$$\Phi = \Phi(R, z) \tag{1}$$

the spatial mass density must possess analogous properties,

$$\delta = \delta(R, z), \tag{2}$$

which is connected with it by Poisson' s equation

$$4\pi G\delta = -\frac{\partial^2\Phi}{\partial R^2} - \frac{1}{R}\frac{\partial\Phi}{\partial R} - \frac{\partial^2\Phi}{\partial z^2}, \tag{3}$$

and also, generally speaking, the phase density

$$\Psi = \Psi(R, z, v_R, v_z, v_\vartheta), \tag{4}$$

which is connected with δ by the corresponding integral equation

$$\delta = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi\, dv_R\, dv_z\, dv_\vartheta. \tag{5}$$

In the general expression for such a phase density, only single-valued axisymmetric integrals of the motion of an individual star may enter as independent arguments,

$$I = I(R, z, v_R, v_z, v_\vartheta), \tag{6}$$

$$\frac{DI}{Dt} = v_R \frac{\partial I}{\partial R} + v_z \frac{\partial I}{\partial z} + \frac{v_\vartheta}{R} \left(v_\vartheta \frac{\partial I}{\partial v_R} - v_R \frac{\partial I}{\partial v_\vartheta} \right) + \frac{\partial \Phi}{\partial R} \frac{\partial I}{\partial v_R} + \frac{\partial \Phi}{\partial z} \frac{\partial I}{\partial v_z} = 0. \tag{7}$$

Such integrals of motion are: the energy integral

$$I_1 = \tfrac{1}{2} v^2 - \Phi(R, z), \tag{8}$$

and the integral of angular momentum with respect to the axis of symmetry

$$I_2 = R v_\vartheta. \tag{9}$$

and the third single-valued axisymmetric integral of motion,

$$I_3 = I_3(R, z, v_R, v_z, v_\vartheta), \tag{10}$$

which, in the case of spherical symmetry of the potential,

$$\Phi = \Phi(r) = \Phi\left(\sqrt{R^2 + z^2}\right) \tag{11}$$

reduces to the spherically symmetric sum of the squares of the integrals of the kinetic moments with respect to the three coordinate axes,

$$I_3^* = (x v_y - y v_x)^2 + (y v_z - z v_y)^2 + (z v_x - x v_z)^2 =$$

$$= r^2 v^2 - (\mathbf{r}\mathbf{v})^2 = r^2 (v^2 - v_r^2) = (R v_z - z v_R)^2 + (R^2 + z^2) v_\vartheta^2, \tag{12}$$

whereas a fourth independent single-valued axisymmetric integral of motion $I_4(R, z, v_R, v_z, v_\vartheta)$ can in fact no longer exist.

Indeed, expressing in it the velocity components in terms of the integrals of motion I_1 (8), I_2 (9), I_3 (10), and expanding it in a Maclaurin power series with respect to I_2, we obtain

$$I_4 = I_4(R, z, I_1, I_2, I_3) = \sum_{k=0}^{\infty} \psi_k(R, z, I_1, I_3)\, I_2^k. \tag{13}$$

If at the initial instant, for some star, $v_\vartheta = 0$, then this condition will be preserved subsequently as well, $I_2 = 0$, and its orbit must lie entirely in the initial

plane $\vartheta = \vartheta_0$, with
$I_4(R, z, I_1, 0, I_3) = \psi_0(R, z, I_1, I_3)$. Any plane orbit determined by specifying three independent single-valued integrals of motion I_1, I_3, and I_4 (the integral of motion I_2 in the case under consideration becomes identically zero) must be one-dimensional or (provided that the initial velocity is less than the corresponding parabolic escape velocity from the system) closed. However, judging from the known particular solutions of the general Bertrand problem ([2],[3]), there are grounds to suppose that all bounded plane orbits can be closed only in exceptional cases, never fully realized for stellar systems (a spherically symmetric potential of a point mass or of a homogeneously distributed mass). Consequently, in real stellar systems the function $\psi_0(R, z, I_1, I_3)$, playing the role of an integral of motion for the stars under consideration, must depend on the integrals of motion I_1 and I_3, i.e.,

$$\psi_0(R, z, I_1, I_3) = \varphi_0(I_1, I_3). \tag{14}$$

Applying analogous reasoning to the integral of motion

$$\frac{I_4 - \varphi_0(I_1, I_3)}{I_2} \tag{15}$$

and successively to the other corresponding integrals of motion, we obtain:

$$\psi_k(R, z, I_1, I_3) = \varphi_k(I_1, I_3), \qquad k = 1, 2, \dots, \tag{16}$$

i.e., the arbitrary fourth single-valued axisymmetric integral of motion I_4 (13) under consideration indeed cannot be independent of the first three integrals of motion:

$$I_4 = \sum_{k=0}^{\infty} \varphi_k(I_1, I_3)\, I_2^k. \tag{17}$$

Thus, in the general case of axial symmetry of the stationary potential (1), there can and must exist only one independent ⋯

of the energy integral I_1 (8) and the angular-momentum integral I_2 (9), a single-valued axisymmetric integral of motion I_3 (10), i.e., in the corresponding general expression for the phase density Ψ of the stellar systems under consideration, as independent arguments there can and must enter, contrary to the widely held opinion ([1,4]), not five, but also not two, rather three single-valued axisymmetric integrals of motion of an individual star:

$$\Psi = \Psi(I_1, I_2, I_3) = \Psi\left(\frac{1}{2}v^2 - \Phi(R, z),\ Rv_\vartheta,\ I_3\right). \tag{18}$$

From the kinetic equation (7) it is easy to see that in any integral of motion (6) the parts even and odd with respect to v_ϑ, and the combinations of the quantities v_R and v_z, must themselves be integrals of motion. By virtue of its uniqueness, the third independent integral of motion I_3 (10) must always reduce to one of such parts. Its known particular form I_3^* (12) (for spherical symmetry of the potential) allows one to regard the integral of motion I_3 as even:

$$I_3(R, z, v_R, v_z, -v_\vartheta) = I_3(R, z, v_R, v_z, v_\vartheta), \tag{19}$$

$$I_3(R, z, -v_R, -v_z, v_\vartheta) = I_3(R, z, v_R, v_z, v_\vartheta). \tag{20}$$

The energy integral I_1 (8) is even with respect to all velocity components, while the angular-momentum integral I_2 (9), being odd with respect to v_ϑ, is even with respect to v_R and v_z. As a result, the phase density Ψ (18) must be even with respect to the set of quantities v_R and v_z:

$$\Psi(R, z, -v_R, -v_z, v_\vartheta) = \Psi(R, z, v_R, v_z, v_\vartheta). \tag{21}$$

In this case

$$\begin{aligned} \overline{v_R} &= \frac{1}{\delta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_R \Psi \, dv_R \, dv_z \, dv_\vartheta \equiv 0, \\ \overline{v_z} &= \frac{1}{\delta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_z \Psi \, dv_R \, dv_z \, dv_\vartheta \equiv 0, \end{aligned} \tag{22}$$

i.e., the possible systematic motions of stellar centroids in stationary axisymmetric stellar systems reduce only to differential, generally speaking, rotation about the axis of symmetry with some velocity

$$\overline{v_\vartheta}(R, z) = \frac{1}{\delta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_\vartheta \Psi \, dv_R \, dv_z \, dv_\vartheta \neq 0. \tag{23}$$

Thus, for such real quasi-stationary axisymmetric stellar systems as the Galaxy and regular extragalactic nebulae, the main observational fact of their kinematics—differential axial rotation—is theoretically explained ([1,5,6]).

The presence of three independent single-valued axisymmetric integrals of motion entering as independent arguments into the general expression of the phase density Ψ (18), under axial symmetry of the stationary potential (1), also theoretically explains a second well-known observational fact from the kinematics of our Galaxy—the triaxiality of the distribution of stellar peculiar velocities ([1,7]):

$$\overline{v_R^2} \neq \overline{v_z^2} \neq \overline{(v_\vartheta - \overline{v_\vartheta})^2}. \tag{24}$$

For spherical symmetry of the stationary potential (11), the phase density Ψ (18), which is the general solution of the corresponding kinetic equation, can contain as independent arguments only the spherically symmetric even integrals of motion I_1 (8) and $I_3 = I_2^*$ (12), while its dependence on the odd integral of kinetic momentum I_2 (9), which does not possess spherical symmetry, must automatically disappear; i.e., in stationary spherically symmetric stellar systems with continuous phase density $\Psi(r, \mathbf{v})$, there must be no systematic motions of the stellar centroids whatever (including their axial rotation): $\overline{v_\vartheta} \equiv 0$.

Such nonrotating spherically symmetric systems may be formed from ordinary, not excessively flattened, rotating axisymmetric stellar systems which in the course of evolution have practically lost their rotation after a considerable dissipation of mass ([8]). Therefore it is natural that among field galaxies there is observed a vast number of small, quite round, spherically symmetric galaxies, while elliptical galaxies with moderate true flattening are practically already completely absent; whereas among the statistically younger galaxies in clusters, precisely the latter still predominate ([9]).

Astrophysical Institute
Academy of Sciences of the Kazakh SSR

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Note: Figure translations are in progress. See original paper for figures.

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