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Abstract

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ON THE NUMBER OF INDECOMPOSABLE NETWORKS AND SOME OF THEIR PROPERTIES

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In the works of B. A. Trakhtenbrot ([1,2]) it is shown that every two-terminal network is a superposition (in the sense of replacing an edge by a network) of indecomposable networks*, i.e., the latter constitute a topological basis, by superpositions of whose elements any network can be obtained. In connection with this, the study of indecomposable networks is of great interest. It turns out that for all $n \geq 7$ there exist indecomposable networks with n edges, and, naturally, the question arises of the number of such networks. R. E. Krichevskii ([4]) ineffectively showed that for some sequence $\{n_i\}$ the number $\Phi(n_i)$ of indecomposable networks with n_i edges is bounded below by the function

$$\left(\frac{Cn_i}{\ln^2 n_i}\right)^{n_i}, \tag{1}$$

where the constant C was not estimated. However, finding it is of interest, since the number $\Phi^*(n)$ of all possible nonisomorphic networks with n edges is estimated by

$$\left(\frac{n}{\ln^2 n}\right)^n < \Phi^*(n) < \left(\frac{2n}{e\ln^2 n}\right)^n.$$

The lower estimate can easily be obtained from Theorem 7 of ([5]), and the upper one is contained in ([6]). Moreover, it was unclear whether estimate (1) holds for all n.

In the present work a lower estimate is considered for the number $\Phi(n)$ of nonisomorphic indecomposable networks with n edges. A class is effectively constructed, consisting of $\Phi'(n)$ nonisomorphic indecomposable networks such that

$$\Phi'(n) > \left(\frac{n}{2e\ln^2 n}\right)^n$$

for all sufficiently large n. Clearly, $\Phi(n) \geq \Phi'(n)$. In addition, two criteria for indecomposability of a network are established, making it possible, for certain constructively specified indecomposable networks and classes of networks, to prove indecomposability without enumerating all subgraphs, as the definition of an indecomposable network requires.

§ 1. Let a network S be given. Here and throughout what follows, strong connectedness of a network is assumed.** A subgraph G of the network S is called a **subnetwork** if it has exactly two boundary vertices with the subgraph $S \setminus G$ (the poles of the network S belonging to the subgraph G are always counted among boundary vertices). A subnetwork is called **nontrivial** if it is not an edge and does not coincide with S. A network having no nontrivial subnetworks is called **indecomposable**. (These definitions were introduced by B. A. Trakhtenbrot ([1,2]) and A. V. Kuznetsov ([3]).)

We shall say*** that a vertex A **controls** a vertex B (an edge–

* For all concepts not defined in the present article, we refer to ([1–3]).

** A network is called **strongly connected** if at least one chain passes through every edge.

*** Cf. (1), § 3, item 4.

by an edge β, subgraph $\mathfrak{B}$), if every chain containing the vertex B (the edge β, the subgraph $\mathfrak{B}$) passes through A.

A network is called **simple** if it is not a parallel connection of several networks and each pair of vertices is joined by no more than one edge.

Theorem 1. *For a simple network S, consisting of more than two edges, to be decomposable it is necessary and sufficient that there exist an internal vertex A controlling some other vertex B.*

Proof. The **necessity** is obvious.

Sufficiency. Suppose the contrary. Then the vertex A is completely internal (Definition 5 ([2])), and, by Lemma 3 ([2]), its star Z_A is bypassable (Definition 4 ([2])), i.e., the network $S \setminus Z_A$ is strongly connected. If B is a pole of the network S, then all chains pass through A, and, by Theorem 4 ([2]), S is a Π-network, i.e., decomposable. If, however, B is not a pole of the network S, then Z_B is not wholly contained in Z_A. But in $S \setminus Z_A$ no chain passes through an arbitrary edge from $Z_B \setminus Z_A$, since any such chain would pass (in S) through B without passing through A, which contradicts the condition of the lemma. Consequently, $S \setminus Z_A$ is not strongly connected, contrary to the assumption of the indecomposability of the network S. The theorem is proved.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

Theorem $1'$. *For a simple network S, consisting of more than two edges, to be decomposable it is necessary and sufficient that there exist an internal vertex A controlling an edge β not incident with it.*

Proof is analogous to the proof of Theorem 1.

A system of chains in a network S is called **defining for the vertex** B (edge β, subgraph $\mathfrak{B}$) if the set-theoretic intersection of the chains of the system consists only of this vertex B (edge β, subgraph $\mathfrak{B}$) and the poles of the network S.

Lemma 1. *In order that there exist, for the vertex B (edge β, subgraph $\mathfrak{B}$), a defining system of chains, it is necessary and sufficient that no internal vertex A control B $(\beta, \mathfrak{B})$.*

Proof. The **necessity** is obvious.

Sufficiency. A defining system of chains for B $(\beta, \mathfrak{B})$ is, for example, the totality of all chains passing through this vertex (edge, subgraph). The lemma is proved.

Thus, the existence, for a vertex (edge, subgraph), of a defining system of chains and the existence for them of an internal controlling vertex mutually exclude one another (and similarly their absence).

Theorem 2. *In order that a network S be indecomposable, it is necessary and sufficient that it be simple and that for each vertex (edge) there exist a defining system of chains.* The theorem is the converse of Theorems 1 and $1'$, valid by virtue of Lemma 1.

Fig. 2. Black points are working vertices, light points are auxiliary vertices

Corollary. If, in an indecomposable network, one joins by an edge two vertices not yet joined by an edge and not both poles of the network, then the resulting network will be indecomposable.

§ 2. Let us construct a class of networks with n edges. We shall call a network with poles A and B and with $(k+1)$ "rungs," shown in Fig. 1, a "truss." Let us connect d such trusses in parallel, identifying all poles A with one another and all the poles B among themselves. Number the trusses, and in the first insert a "notch" γ, and then connect them by $(d-1)$ "links" $a_1, a_2, \dots, a_{d-1}$ (Fig. 2). The poles in Fig. 2 are shown schematically by shaded rectangles. In each of the trusses we distinguish "working" and "auxiliary" vertices (Fig. 2). The network shown in Fig. 2 will be called auxiliary. It contains

$$p = [d(2k+6)+1]$$

edges (whence

$$kd = \frac{p-(6d+1)}{2}$$

).

If we now connect, by no more than one edge, an arbitrary set of pairs of working vertices, excluding pairs already connected by an edge and pairs consisting of vertices of one truss, then an indecomposable network will be obtained,* and for different sets of pairs of working vertices the networks will be non-isomorphic. The number of such pairs of working vertices is equal to

$$\frac{kd(kd-k)}{2} = \frac{(kd)^2}{2}\left(1-\frac{1}{d}\right) = \frac{(p-c)^2}{8}\left(1-\frac{1}{d}\right), \quad \text{where } c = 6d+1.$$

In order that the network consist of n edges, it is necessary to insert into the auxiliary network $(n-p)$ edges, i.e. to connect by edges arbitrary $(n-p)$ admissible pairs of working vertices. Then the number $\Phi'(n)$ of distinct non-isomorphic networks constructed by the method described above is equal to

$$\Phi'(n) = C^{n-p}_{\frac{(p-c)^2}{8}\left(1-\frac{1}{d}\right)}. \tag{2}$$

§ 3. **Lemma 2.** *For different sets of pairs of working vertices of the network, the networks constructed in § 2 are non-isomorphic.*

Proof. Take two isomorphic networks and verify that the isomorphism is established uniquely, i.e. one network cannot correspond to two different sets of pairs of working vertices. Indeed, the poles are identified uniquely because of their different degrees. Then chains of length 2 are distinguished; they are the bases of the trusses, which, in turn, are ordered uniquely by the "links," since the first truss contains the notch. The working vertices of each truss are ordered from entrance to exit uniquely, whence trusses with identical numbers in isomorphic networks are identified uniquely; consequently, this can be said of both networks as well. The lemma is proved.

Lemma 3. *The networks constructed in § 2 are indecomposable.*

Fig. 3

Fig. 4. I —first chain, II —second chain, III —third chain

Proof. First we shall prove the indecomposability of the auxiliary network. First, it is easily verified that the network is simple. Second, for each vertex there exists a determining system of chains. Indeed, in the auxiliary network there are three types of vertices (Fig. 3). For the 1st, 2nd, and 3rd types the desired system consists respectively of two (Fig. 4a), three (Fig. 4b), and three (Fig. 4c) chains. Consequently, by Theorem 2, the auxiliary network is indecomposable. Hence, by the corollary to Theorem 2, the whole network is also indecomposable. The lemma is proved.

* For the proof see § 3, Lemmas 2 and 3.

§ 4. **Theorem 3.** $\Phi(n) > \left(\frac{n}{2e\ln^2 n}\right)^n$ for sufficiently large n.

Proof. We shall find a lower estimate for $\Phi'(n)$. Denote

$$\frac{(p-c)^2}{8}\left(1-\frac{1}{d}\right) = a, \qquad n-p = b.$$

From (2), taking $b^2 < a^2 < b^2(a-b)$:

$$\Phi'(n) = C_a^b = \frac{a!}{b!(a-b)!} > \frac{a^a}{b^b(a-b)^{a-b}}\,\frac{\sqrt{2\pi a}\,e^{-\frac{\theta a}{12b(a-b)}}}{2\pi\sqrt{b(a-b)}} > \left(\frac{a}{b}\right)^b \frac{1}{(1-b/a)^{a-b}\sqrt{2\pi b}}. \tag{3}$$

Transform and estimate from below

$$\frac{1}{(1-b/a)^{a-b}}.$$

$$\frac{1}{(1-b/a)^{a-b}} = \left(1-\frac{b}{a}\right)^{b-a} = \left(1-\frac{b}{a}\right)^{\frac{a}{b}\left[\frac{b}{a}(b-a)\right]} > e^{b\left(1-\frac{b}{a}\right)}.$$

Then

$$\Phi'(n) > \left(\frac{ae^{\left(1-\frac{b}{a}\right)}(2\pi b)^{-\frac{1}{2b}}}{b}\right)^b. \tag{4}$$

Let

$$d = [\ln^2 n], \qquad k = \left[\frac{\frac{2n}{\ln n}-c}{2\ln^2 n}\right]$$

(then

$$\frac{2n}{\ln n} > p > \frac{2n}{\ln n} - 2\ln^2 n$$

and (3) is satisfied). Substituting in (4):

$$\Phi'(n) > \left[\frac{n\left(1-\frac{c\ln n}{2n}-\frac{\ln^3 n}{n}\right)^2\left(1-\frac{1}{d}\right)e^{\left(1-\frac{b}{a}\right)}(2\pi b)^{-\frac{1}{2b}}}{2\ln^2 n\left(1-\frac{2}{\ln n}+\frac{2\ln^2 n}{n}\right)}\right]^{n\left(1-\frac{2}{\ln n}\right)} = Z^n,$$

$$\ln Z = \left(1-\frac{2}{\ln n}\right)\left[\ln n + 2\ln\left(1-\frac{c\ln n}{2n}-\frac{\ln^3 n}{n}\right) + \ln\left(1-\frac{1}{d}\right)\right.$$

$$+1 - \frac{b}{a} - \frac{\ln 2\pi b}{2b} - \ln 2 - 2\ln\ln n - \ln\left(1 - \frac{2}{\ln n} + \frac{2\ln^2 n}{n}\right)\Bigg]$$

$$= \ln n - 2\ln\ln n + 1 - \ln 2 - 2 + \frac{4\ln\ln n}{\ln n} + O\left(\frac{1}{\ln n}\right),$$

$$Z = \frac{n}{2e\ln^2 n}\, e^{\frac{4\ln\ln n}{\ln n} + O\left(\frac{1}{\ln n}\right)}.$$

Since the exponent of e is positive for sufficiently large n, for such n

$$Z > \frac{n}{2e\ln^2 n},$$

whence

$$\Phi'(n) > Z^n > \left(\frac{n}{2e\ln^2 n}\right)^n,$$

and the theorem is proved, since $\Phi(n) \geq \Phi'(n)$.

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Note: Figure translations are in progress. See original paper for figures.