# ON ENTIRE FUNCTIONS OF FINITE DEGREE AND ON FUNCTIONS OF COMPLETELY REGULAR GROWTH OF SEVERAL VARIABLES

1958

SovietRxiv

**Abstract**

**Full Text**

**MATHEMATICS**

**L. I. RONKIN**

# ON ENTIRE FUNCTIONS OF FINITE DEGREE AND ON FUNCTIONS OF COMPLETELY REGULAR GROWTH OF SEVERAL VARIABLES

*(Presented by Academician S. N. Bernstein on 25 X 1957)*

An entire function $f(z_1, z_2, \dots, z_n)$, where $z_k = x_k + iy_k$, will be called a **function of finite degree with respect to the aggregate of variables** $z_1, z_2, \dots, z_n$, if there exist two constants $A > 0$ and $a > 0$ such that, for arbitrary $z_1, z_2, \dots, z_n$, the inequality

$$|f(z_1, \dots, z_n)| < A \exp[a(|z_1| + |z_2| + \dots + |z_n|)].$$

holds. Let $f(u, z_1, z_2, \dots, z_n)$ be an entire function of finite degree with respect to the aggregate of variables $u, z_1, \dots, z_n$. Consider this function for fixed values $z_1, \dots, z_n$; we obtain an entire function of finite degree of the single variable $u$. The degree of such a function and its indicator of growth will, obviously, depend on the fixed values $z_1, z_2, \dots, z_n$. Denote the degree of the function by $\sigma_{f,u}(z_1, z_2, \dots, z_n)$, and the growth indicator by $h_{f,u}(\theta, z_1, \dots, z_n)$. According to the definition of degree and growth indicator for functions of one variable ([1]), we shall have

$$\sigma_{f,u}(z_1, z_2, \dots, z_n) = \lim_{r\to\infty} \frac{1}{r} \max_{|u|=r} \ln |f(u, z_1, \dots, z_n)|,$$

$$h_{f,u}(\theta, z_1, \dots, z_n) = \lim_{r\to\infty} \frac{1}{r} \ln |f(re^{i\theta}, z_1, z_2, \dots, z_n)|.$$

We also denote

$$\bar{\sigma}_{f,u}(\theta) = \max_{z_1, z_2, \dots, z_n} \sigma_{f,u}(z_1, z_2, \dots, z_n), \qquad \bar{h}_{f,u}(\theta) = \max_{z_1, z_2, \dots, z_n} h_{f,u}(\theta, z_1, z_2, \dots, z_n).$$

In the case of functions of two variables, we previously ([2]) proved the following theorem*:

**Theorem 1.** Let $f(u, z)$ be an entire function of finite degree with respect to the aggregate of variables $u, z$. Then the set $M_\varepsilon$ of points $z$ at which the inequality

$$\sigma_{f,u}(z) \leqslant \bar{\sigma}_{f,u} - \varepsilon$$

is satisfied, satisfies the condition:

A. $M_\varepsilon = \sum_{n=1}^{\infty} F_n$, where $F_n$ is closed, $F_n \subset F_{n+1}$, and each $F_n$ has absolute measure zero.

In the case of a larger number of variables, Theorem 1 immediately implies:

* In ([2]) a somewhat stronger theorem was proved. Theorems 2 and 3 could also have been strengthened, but

**Theorem 2.** Let $f(u, z_1, \ldots, z_n)$ be an entire function of finite degree in the aggregate of the variables $u, z_1, \ldots, z_n$. Then the set $M_\varepsilon$ of points $z_1, z_2, \ldots, z_n$ at which the inequality

$$\sigma_{f,u}(z_1, \ldots, z_n) \leq \overline{\sigma}_{f,u} - \varepsilon$$

is satisfied has the following property:

**B.** The intersection of $M_\varepsilon$ with any two-dimensional analytic plane of the form $z_k = a_k w + b_k,\ k = 1, 2, \ldots, n$, satisfies condition A, except in those cases when the plane under consideration lies entirely in $M_\varepsilon$.

For $h_{f,u}(\theta, z_1, \ldots, z_n)$ the corresponding theorem may be formulated as follows:

**Theorem 3.** The set $M_\varepsilon(\theta)$ of points $z_1, z_2, \ldots, z_n$ at which, for fixed $\theta$, the inequality

$$h_{f,u}(\theta, z_1, z_2, \ldots, z_n) \leq \overline{h}_{f,u}(\theta) - \varepsilon$$

is satisfied has property B.

Pólya and Plancherel ([3]) introduced the indicator of growth $h_f(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in the aggregate of variables, which is defined by the equality

$$h_f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sup_{x_1, x_2, \ldots, x_n} h_f(\lambda_1, \lambda_2, \ldots, \lambda_n;\, x_1, x_2, \ldots, x_n),$$

where

$$h_f(\lambda_1, \lambda_2, \dots, \lambda_n;\, x_1, x_2, \dots, x_n) = \overline{\lim} \frac{1}{r} \left| f(x_1 - i\lambda_1 r, x_2 - i\lambda_2 r, \dots, x_n - i\lambda_n r) \right|,$$

$x_k$ and $\lambda_k$ are real numbers and

$$\sum_{k=1}^{n} \lambda_k^2 = 1.$$

**Theorem 4.** Let $f(z_1, z_2, \dots, z_n)$ be an entire function of finite degree in the aggregate of variables. Then the set $M'_\varepsilon$ of points $x_1, x_2, \dots, x_n$ at which

$$h_f(\lambda_1, \lambda_2, \dots, \lambda_n;\, x_1, x_2, \dots, x_n) \le h_f(\lambda_1, \lambda_2, \dots, \lambda_n) - \varepsilon$$

is satisfied has $\operatorname{mes}^{(n)} M'_\varepsilon = 0$ (here, as below, $\operatorname{mes}^{(k)} E$ denotes the Lebesgue measure of the set $E$ in $k$-dimensional space).

The proof of this theorem is based on Theorem 3 and the following lemma.

**Lemma 1.** Let a set $E$ in the $n$-dimensional space $R_n$ of real variables $x_1, x_2, \dots, x_n$ satisfy the condition:

**C.** The intersection of the set $E$ with any line of the form
$x_i = a_i t + b_i,\ i = 1, 2, \dots, n$, with real $a_i, b_i$, either has one-dimensional Lebesgue measure zero, or this line belongs entirely to $E$.

Then, if $\operatorname{mes}^{(n)} E > 0$, then $E = R_n$.

We omit the proof of the lemma.

Let us outline the proof of Theorem 4. For this purpose introduce the function

$$g(u, z_1, z_2, \dots, z_n) = f(z_1 - i\lambda_1 u, z_2 - i\lambda_2 u, \dots, z_n - i\lambda_n u),$$

which is an entire function of finite degree in the aggregate of the variables $u, z_1, z_2, \dots, z_n$. Clearly,

$$h_{g,u}(0, x_1, x_2, \dots, x_n) = h_f(\lambda_1, \lambda_2, \dots, \lambda_n;\, x_1, x_2, \dots, x_n).$$

We shall show that $h_f(\lambda, \lambda_2, \dots, \lambda_n) > \overline{h}_{g,u}(0) - \varepsilon$, or, what is the same,

$$\sup_{x_1, x_2, \dots, x_n} h_{g,u}(0, x_1, x_2, \dots, x_n) > \sup_{z_1, z_2, \dots, z_n} h_{g,u}(0, z_1, z_2, \dots, z_n) - \varepsilon.$$

Indeed, if for real $a_i$ and $b_i$

$$\sup_{\left\{\begin{array}{c} x_i = a_i t + b_i \\ i = 1, 2, \dots, n \end{array}\right\}} h_{g,u}(0, x_1, x_2, \dots, x_n) < \overline{h}_{g,u}(0) - \varepsilon,$$

then, by virtue of Theorem 3 and the fact that the intersection of a set of absolute harmonic measure zero with any line has linear Lebesgue measure zero ([4]), we obtain that the analytic plane $z_i = a_{iw} + b_i,\ i = 1, 2, \dots, n$, intersecting $R_n$ along the line $x_i = a_{it} + b_i,\ i = 1, 2, \dots, n$, belongs entirely to $M_\varepsilon(0)$. But, as is easy to show, the totality of analytic planes of the form $z_i = a_{iw} + b_i,\ i = 1, 2, \dots, n$, where $a_i$ and $b_i$ are real numbers, exhausts the whole space $Z_n$ of the complex variables $z_1, z_2, \dots, z_n$, i.e. $M_\varepsilon(0) = Z_n$, which is obviously impossible.

From the assertion proved it follows that $h_f(\lambda_1, \lambda_2, \dots, \lambda_n) = h_{g,u}(0)$, and $M_\varepsilon(0) \supset M'_\varepsilon$, where $M'_\varepsilon$ is the set appearing in the condition of Theorem 4. Hence, applying Theorem 3, we obtain that the set $M'_\varepsilon$ has property C. From this, by virtue of Lemma 1, we obtain that if $\text{mes}^{(n)} M'_\varepsilon > 0$, then $M'_\varepsilon = R_n$, and consequently $R_n \subset M_\varepsilon(0)$, which is impossible by what has been proved*. Thus Theorem 4 is completely proved.

The theorems proved above make it possible to define in a natural way, for functions of many variables, a class of functions of completely regular growth.

For the case of functions of one variable this class of functions was first defined and studied in detail by B. Ya. Levin ([1]). Following B. Ya. Levin, an entire function $F(z)$ of finite degree** is called a **function of completely regular growth** if for every $\theta \in [0, 2\pi]$ there exists the limit

$$h_f(\theta) = \lim \frac{\ln |F(re^{i\theta})|}{r},$$

under the condition that $r$ tends to infinity, assuming all positive values except, perhaps, a set of zero relative measure. The functions of completely regular growth of one variable include, in particular, functions of finite degree bounded on the real axis. Among the properties of functions of completely regular growth one should note, as one of the most important, the following:

$$h_f(\theta) + h_g(\theta) = h_{f \cdot g}(\theta),$$

where $f(z)$ is a function of completely regular growth; $g(z)$ is an entire function of finite degree.

In the case of functions of several variables, we shall call a function $F(z_1, z_2, \dots, z_n)$ a **function of completely regular growth** if, for any real $\lambda_1, \lambda_2, \dots, \lambda_n$ with

$$\sum_{k=1}^{n} \lambda_k^2 = 1$$

and for almost all, in the sense of Lebesgue measure, real $x_1, x_2, \dots, x_n$, there exists the limit

$$h_F(\lambda_1, \lambda_2, \dots, \lambda_n) = \lim \frac{1}{r} \ln |F(x_1 - i\lambda_1 r,\ x_2 - i\lambda_2 r, \dots, x_n - i\lambda_{nr})|,$$

where $r$ tends to infinity, assuming all positive values except, perhaps, a set of zero relative measure.

With this definition of the class of functions of completely regular growth, this class will contain, for example, entire functions of finite degree bounded on $R_n$. The proof of this fact follows from Theorem 4 and from the properties of functions of completely regular growth of one variable. Using earlier published results of the author ([5]), one can prove the following stronger theorem.

**Theorem 5.** If a function $f(z_1, z_2, \dots, z_n)$ of finite degree in the aggregate of variables satisfies the condition

$$|f(x_1, x_2, \dots, x_n)| \leqslant \alpha_1(|x_1|)\alpha_2(|x_2|) \cdots \alpha_n(|x_n|),$$

* By analogous arguments one can prove that $\operatorname{mes}^{(2n)} M_\varepsilon = 0$ and $\operatorname{mes}^{(2n)} M_\varepsilon(\theta) = 0$, where $M_\varepsilon$ and $M_\varepsilon(\theta)$ are the sets appearing in the conditions of Theorems 2 and 3.

** The definition is given for functions of any specified order.

where the functions $\alpha_i(t) > 0$ are such that

$$\int_0^\infty \frac{\ln \alpha(t)}{1 + t^2}\, dt < \infty$$

and either $\alpha_i(t_2) \geq \alpha_i(t_1)$ for $t_2 > t_1 > 0$, or $\alpha_i(t_1 + t_2) \leq \alpha_i(t_1) \cdot \alpha_i(t_2)$, then $f(z_1, z_2, \dots, z_n)$ is a function of completely regular growth.

Functions of completely regular growth of several variables, defined by the method indicated above, possess a number of properties analogous to the properties of functions of completely regular growth of one variable. Thus, for example, it is not difficult to show that if $f(z_1, z_2, \dots, z_n)$ is of completely regular growth, and $g(z_1, z_2, \dots, z_n)$ is of finite degree, then

$$h_{fg}(\lambda_1, \lambda_2, \dots, \lambda_n) = h_f(\lambda_1, \lambda_2, \dots, \lambda_n) + h_g(\lambda_1, \lambda_2, \dots, \lambda_n).$$

In conclusion we note that, with the aid of functions of completely regular growth, one can prove, for the case of many variables, a generalization of the well-known Titchmarsh theorem on convolution. In this case the following theorem is obtained*.

**Theorem 6.** Let, for finite functions $f$ (i.e. functions equal to zero outside some domain), $K_f(\lambda_1, \lambda_2, \dots, \lambda_n)$ denote the support function of the smallest convex domain outside which $f(x_1, x_2, \dots, x_n) = 0$. Then the equality

$$K_f(\lambda_1, \lambda_2, \dots, \lambda_n) + K_g(\lambda_1, \lambda_2, \dots, \lambda_n) = K_{f\times g}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

holds, where $f$ and $g$ are finite functions belonging to $\mathcal{L}_1$, and

$$f \times g = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1 - t_1,\ x_2 - t_2, \dots, x_n - t_n) g(t_1, t_2, \dots, t_n)\, dt_1\, dt_2 \cdots dt_n.$$

For the proof of this theorem it is enough to observe that if $f(z_1, z_2, \dots, z_n)$ is an entire function of finite degree, $f(x_1, x_2, \dots, x_n) \in \mathcal{L}_2$, and

$$F(t_1, t_2, \dots, t_n) =$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-i(t_1 x_1 + t_2 x_2 + \cdots + t_{nx} n)] f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n,$$

then, as was proved by Pólya and Plancherel ([3]), $F(x_1, x_2, \dots, x_n)$ is a finite function and $K_f(\lambda_1, \lambda_2, \dots, \lambda_n) = h_1(\lambda_1, \lambda_2, \dots, \lambda_n)$, after which the theorem is proved by a literal repetition of the arguments carried out by B. Ya. Levin in his proof of the analogous theorem for the case of one variable.

Received
24 X 1957

## REFERENCES

1. B. Ya. Levin, *Distribution of Zeros of Entire Functions*, 1953.
2. L. I. Ronkin, Mat. sborn., 39 (81), 2, 253 (1956).
3. M. Plancherel, G. Polya, Comm. Math. Helv., 10, 110 (1937).
4. R. Nevanlinna, *Eindeutige analytische Funktionen*, M.–L., 1941.

5. L. I. Ronkin, DAN, 92, No. 5 (1953).

6. J. Mikusinski, C. Ryll-Nardzewski, Studia Math., 13, No. 1 (1953).

* In a somewhat different form and by other methods, a generalization of Titchmarsh' s theorem was given in 1953 by Mikusinski and Ryll-Nardzewski ([6]). The theorem of Mikusinski and Ryll-Nardzewski can be obtained as a simple consequence of Theorem 6. Likewise, Theorem 6 can be obtained from the theorem of Mikusinski and Ryll-Nardzewski by simple arguments.

*Note: Figure translations are in progress. See original paper for figures.*