



---

Soviet-era science, translated into English

# Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-195801.39848>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

## Reports of the Academy of Sciences of the USSR

1958. Volume 120, No. 3

**MATHEMATICS**

**I. V. GELFAND**

### ON A NONLINEAR OPERATOR

*(Presented by Academician V. I. Smirnov, January 17, 1958)*

1. Let  $M(u)$  and  $N(v)$  be two mutually complementary  $N'$ -functions, and let  $L_M^*(\Omega)$  and  $L_N^*(\Omega)$  be the corresponding Orlicz spaces ( $\Omega$  is an  $n$ -dimensional bounded domain with boundary  $\Gamma$ ) <sup>(1)</sup>, where the function  $M(u)$  is such that

$$\frac{\int_{\Omega} M(u) d\Omega}{\|u\|_M} \rightarrow \infty \quad \text{as } \|u\|_M \rightarrow \infty \quad (1)$$

(where  $\| \cdot \|_M$  denotes the norm in the space  $L_M^*(\Omega)$ )\*.

Suppose that a functional is given,

$$j(u) = \int_{\Omega} f \left( x_1, \dots, x_n; u(x); \dots \frac{\partial^l u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right) d\Omega, \quad (2)$$

where

$$f \geq m M(D^{(l)}u) \quad \left( m > 0 \text{ is a constant; } D^{(l)}u = \sum_{i_1 \dots i_n} \left| \frac{\partial^l u}{\partial x_{i_1} \dots \partial x_{i_n}} \right| \right), \quad (3)$$

\end{equation}

$$\sum_{k,s=0}^l \frac{\partial^2 f}{\partial u_{k_1, \dots, k_n}^{(k)} \partial u_{s_1, \dots, s_n}^{(s)}} t_{k_1, \dots, k_n}^{(k)} t_{s_1, \dots, s_n}^{(s)} > 0 \quad \left( \sum_{k=0}^l |t_{k_1, \dots, k_n}^{(k)}|^2 \neq 0 \right) \quad (4)$$

(just as is done below, one may also consider the case in which, under the sign of  $f$ , instead of the function  $u(x)$  there stands a vector-valued function  $\mathbf{u} = (u_1, \dots, u_R)$ ).

In the present note we consider the problem of minimizing the functional

$$j_h(u) = j(u) - \int_{\Omega} uh \, d\Omega, \quad h \in L_N^*(\Omega), \quad (5)$$

for which the class of admissible functions  $\Phi$  (assumed nonempty) is defined by the condition  $j(u) < +\infty$  and by the boundary conditions\*\*

$$\frac{\partial^k u}{\partial x_1^{b_1} \dots \partial x_n^{b_n}} \Big|_{\Gamma} = 0 \quad (k = 0, 1, \dots, l - 1). \quad (6)$$

The solution of this problem is given in § 2.

\* As M. A. Krasnosel'skii kindly informed me, Ya. B. Rutitskii constructed an example from which it can be seen that relation (1) is not fulfilled in any Orlicz space defined by a function complementary to a function satisfying the  $\Delta^2$ -condition. Concerning the  $\Delta^2$ -condition, see (2).

\*\* The homogeneity of the boundary conditions does not diminish the generality.

If the Euler-Lagrange equation for the functional (2) has the form

$$Lu = 0, \quad (7)$$

then to the variational problem (5)–(6) there corresponds the Euler-Lagrange equation

$$Lu = h \quad (7')$$

with boundary conditions (6). We shall call the solution of the variational problem (5)–(6) the generalized solution of problem (7')–(6). Considering it as an operator  $u = u[h]$  acting on the right-hand side of equation (7'), in §§ 3–5 we establish the boundedness and certain sufficient conditions for complete continuity of this operator.

**2. Theorem 1.** *Whatever the function  $h \in L_N^*(\Omega)$ , the problem of minimizing the functional  $j_h(u)$  has a unique solution in the class  $\Phi$ .*

**Proof.** By conditions (1), (3) and the generalized Hölder inequality (1), for fixed  $h \in L_N^*(\Omega)$  the functional  $j_h(u)$  is increasing. With the aid of the embedding theorems and the semicontinuity theorem established in paper (3)\*, we conclude that  $j_h(u)$  is (o)-weakly lower semicontinuous and has a finite lower bound on  $\Phi$ . Hence follows the existence of a solution  $u = u[h]$  of problem (5)–(6). Its uniqueness follows from condition (4).

Let  $F_M^{(l)}$  be the space of functions all of whose  $l$ -th generalized derivatives belong to  $L_M^*(\Omega)^{**}$ . Then any minimizing sequence converges to the solution  $u[h]$  ( $o$ )-weakly in  $F_M^{(l)}$ , i.e. the higher derivatives converge ( $o$ )-weakly in  $L_M^*(\Omega)$ , and the lower ones converge in the norm of those Orlicz spaces in which they are contained by the embedding theorems.

**3. Theorem 2.**  $u[h]$  is a bounded operator (as an operator from  $L_N^*(\Omega)$  into  $F_M^{(l)}$ ).

Indeed, otherwise there would exist a sequence  $h_n \in L_N^*(\Omega)$  such that  $\|h_n\|_N \leq A$  ( $n = 1, 2, \dots$ ), while  $\|u_n\|_{F_M^{(l)}} = \|u[h_n]\|_{F_M^{(l)}} \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $h_0 \in L_N^*(\Omega)$  be some ( $o$ )-weak limit of the sequence  $\{h_n\}$ , and let  $u_0 = u[h_0]$ . By (1),  $j_{h_n}(u_n) \rightarrow \infty$  as  $n \rightarrow \infty$ ; hence

$$j_{h_n}(u_0) - j_{h_0}(u_0) > j_{h_n}(u_n) - j_{h_0}(u_0) \rightarrow \infty$$

as  $n \rightarrow \infty$ . On the other hand, from the results of paper (3) it follows that  $u_0 \in E_M(\Omega)^{***}$ , and therefore

$$j_{h_n}(u_0) - j_{h_0}(u_0) = \int_{\Omega} u_0(h_0 - h_n) d\Omega \rightarrow 0$$

as  $n \rightarrow \infty$ .

**4. Lemma.** If  $h_n \rightarrow h_0$  ( $o$ )-weakly, then  $u_n = u[h_n]$  is a minimizing sequence for the functional  $j_{h_0}(u)$ .

**Proof.** Let  $u_0 = u[h_0]$ . From the inequalities

$$j_{h_n}(u_n) - j_{h_0}(u_0) < \int_{\Omega} u_0(h_0 - h_n) d\Omega, \quad j_{h_0}(u_0) - j_{h_n}(u_n) < \int_{\Omega} u_n(h_n - h_0) d\Omega$$

there follows the estimate

$$\left| j_{h_n}(u_n) - j_{h_0}(u_0) \right| \leq \left| \int_{\Omega} (h_n - h_0)u_0 d\Omega \right| + \left| \int_{\Omega} (h_n - h_0)u_n d\Omega \right|,$$

which leads to the estimate

$$\left| j_{h_0}(u_n) - j_{h_0}(u_0) \right| \leq \left| \int_{\Omega} (h_n - h_0)u_0 d\Omega \right| + 2 \left| \int_{\Omega} (h_n - h_0)u_n d\Omega \right|. \quad (8)$$

\* We assume the domain  $\Omega$  and boundary  $\Gamma$  satisfy the conditions of these theorems.

\*\* For this space see also (3).

\*\*\* As usual,  $E_M(\Omega)$  denotes the closure in the norm  $\|\cdot\|_M$  of the set of bounded functions.

the first term on the right-hand side of which tends to zero as  $n \rightarrow \infty$ . By Theorem 2,  $\|u_n\|_{F_M^{(l)}} \leq A_1$  ( $n = 1, 2, \dots$ ). Starting from the results of [3], one can see that every sequence bounded in  $F_M^{(l)}$  is compact in  $E_M(\Omega)$ . Let  $u_{n_k} \rightarrow u' \in E_M(\Omega)$  as  $k \rightarrow \infty$  in the norm  $\|\cdot\|_M$ . It is easily calculated that

$$\int_{\Omega} (h_{n_k} - h_0)u_{n_k} d\Omega \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It then follows from (8) that  $\{u_{n_k}\}$  is a minimizing sequence for the functional  $j_{h_0}(u)$ , so that  $u' = u_0$  (Sec. 2). Hence it is no longer difficult to conclude that the entire sequence  $\{u_n\}$  is a minimizing sequence for the functional  $j_{h_0}(u)$ .

From the lemma proved and the remark at the end of Sec. 2 it follows:

**Theorem 3.** The operator  $u[h]$  is  $(o)$ -weakly continuous, i.e., if  $h_n \rightarrow h_0$   $(o)$ -weakly in  $L_N^*(\Omega)$ , then  $u_n \rightarrow u_0$   $(o)$ -weakly in  $F_M^{(l)}$ .

**5. Theorem 4.** Suppose  $M(u) > |u|^2$ ,

$$\sum_{k,s=0}^l \frac{\partial^2 f}{\partial u_{k_1, \dots, k_n}^{(k)} \partial u_{s_1, \dots, s_n}^{(s)}} t_{k_1, \dots, k_n}^{(k)} t_{s_1, \dots, s_n}^{(s)} \geq m_1 \sum_{k=0}^l |t_{k_1, \dots, k_n}^{(k)}|^2 \quad (4')$$

$$(m_1 > 0 - \text{constant}),$$

at every point of the class  $\Phi$ ; the functional  $j(u)$  has a Gâteaux differential and the basic assumptions of Sec. 1 are satisfied. If the  $N'$ -function  $M_1(u)$  grows essentially more slowly than  $M(u)^*$ , then the operator  $u[h]$ , considered as an operator from  $L_N^*(\Omega)$  into  $F_{M_1}^l$ , carries every  $(o)$ -weakly convergent sequence into a norm-convergent one.

The proof is not difficult to obtain, relying on the following considerations: by virtue of (4'),  $D^{(l)}u_n \rightarrow D^{(l)}u_0$  in the norm of the space  $L_2$ , and hence also in measure; moreover, the norms  $\{\|D^{(l)}u_n\|_M\}$  ( $n = 1, 2, \dots$ ) are uniformly bounded (the notation is the same as in Secs. 3, 4); finally, the lemma of Sec. 4 is valid.

Leningrad State Pedagogical Institute  
named after A. I. Herzen

Received  
16 I 1958

## References

1. M. A. Krasnosel'skii, Ya. B. Rutitskii, *Proceedings of the Seminar on Functional Analysis*, Voronezh State University, no. 1 (1956).

2. M. A. Krasnosel' skii, Ya. B. Rutitskii, DAN, 89, No. 4 (1953).

3. I. V. Gel' man, DAN, 119, No. 4 (1958).

\* That is,  $M(u)/M_1(\lambda u) \rightarrow +\infty$  as  $u \rightarrow +\infty$ , whatever the number  $\lambda$ .

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*