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MATHEMATICS

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ON THE LOCAL NILPOTENCE OF LIE RINGS SATISFYING THE ENGEL CONDITION

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As is known, the question of the local nilpotence of Lie rings with the n-th Engel condition has so far been solved only for particular values $n \leq 7$ ([1]). The purpose of the present note is to give a brief exposition of the proof of the following theorem.

Theorem A. An arbitrary Lie ring L satisfying the n-th Engel condition

$$[uv^n] = [\dots[[uv]v]\dots v] = 0$$

and having characteristic zero or finite characteristic $p > n + [n/2]$, is locally nilpotent.

By the characteristic of a ring is meant here the common order of all nonzero elements in its additive group (the prime number p). The assertion of Theorem A is in fact true for a Lie ring L in which, from the equality $mu = 0,\ u \in L,\ m \leq n + [n/2]$, it follows that $u = 0$.

We note that the main idea of the proof of Theorem A is stated explicitly in § 5 of ([1]). The proof is based on several auxiliary propositions, among which Lemma 1 occupies the most important place. The assertion contained in it may be formulated as follows: the local nilpotence of a Lie ring L with the n-th Engel condition and characteristic $p > n$ will be proved if one establishes the existence in it of an element $c \neq 0$ for which the identity

$$[cu^\alpha c] = 0, \quad \alpha = 1, 2, 3,$$

holds for every $u \in L$. Such an element c can be found only when $p > n + [n/2]$; however, it is evident that the difficulties remaining for arbitrary $p > n$, even if they are not purely technical, are in any case such that attempts to overcome them are hopeless.

Keeping essentially the notation of ([1]), let us introduce certain simplifications essential for the subsequent exposition. We shall everywhere regard as established the local nilpotence of Lie rings with the $(n-1)$-st Engel condition and a given characteristic. Assuming a Lie ring L, satisfying the n-th Engel condition, to be locally regular (and wishing thereby to arrive at a contradiction), we may, using the passage to the quotient ring modulo the radical N, always suppose that $N(L) = 0$. In particular, L will be a ring without center, and therefore isomorphic to the ring D_L of inner derivations. We shall simply identify L and D_L, agreeing to understand everywhere by a Lie ring a set of elements (denoted by small Latin letters $a, b, \ldots$) closed with respect to the operations of addition and "bracket" multiplication

$$[uv] = uv - vu.$$

The elements of the enveloping ring $\mathfrak{A}_L$ will be denoted by capital Latin letters $A, B, \ldots$. If

$$A = a_1 a_2 \ldots a_m = 0,$$

then this means that

$$[uA] = [\ldots[[ua_1]a_2]\ldots a_m] = [ua_1 a_2 \ldots a_m] = 0$$

for every $u \in L$. Conversely, from the relation

$$[ua_1 a_2 \ldots a_m] = 0,$$

identically in u, it follows that

$$A = a_1 a_2 \ldots a_m = 0.$$

The ring L satisfies the n-th Engel condition when $u^n = 0$ for all $u \in L$. If the characteristic of the ring $p > n$, then for

for an arbitrary pair of elements $u, v \in L$ the identities

$$\sum_{0 \le i < n-1} v^i u v^{n-1-i} = 0, \tag{1}$$

$$\sum_{0 \le i < n-1} (-1)^{i+1} \binom{n}{i+1} v^i u v^{n-1-i} = 0. \tag{2}$$

Definition. Let us denote by $c_{(m)}$ a nonzero element of a Lie ring satisfying the identities

$$c_{(m)} u^{\alpha} c_{(m)} = 0, \qquad \alpha = 0, 1, 2, \ldots, 2m-1 \quad (m \ge 1).$$

It turns out that the following fact holds.

Lemma 1. *From the existence of an element $c_{(2)}$ in a Lie ring L with the n-th Engel condition and of characteristic $p > n$ (or $p = 0$), its local nilpotency follows.*

The proof of this lemma is rather cumbersome, though not difficult. It is readily verified that the element

$$c'_{(m)} = [c_{(m)} a^{2m+1} c_{(m)}], \qquad m \geq 1,$$

also has the properties of the element $c_{(m)}$. But for $m > 1$ a much stronger assertion is valid, namely that

$$c'_{(m)} u^{2m} c'_{(m)} v^{2m} c'_{(m)} = 0,$$

whatever elements a, u, v of L we take. After this it is already comparatively easy to show that the right-normed product

$$[c'_{(m)} b^{2m+1} c'_{(m)}], \quad b \in L,$$

may be taken as the element $c_{(m+1)}$. It is assumed that $2m+5 \leq p$. Performing a chain of successive transitions from $c_{(m)}$ to $c_{(m+1)}$, we then consider in L the ideal $\mathfrak{N}$ generated by the element $c_{(m_0)}$ with index

$$m_0 = [(n-1)/2].$$

The latter is possible, since $n < p$, and hence $2m_0 + 3 \leq p$.

With the aid of identity (1), the elements $c_{(m_0)} u^{n-2} c_{(m_0)}$ and $c_{(m_0)} u^{n-1} c_{(m_0)}$ can be expressed in terms of products $A \cdot B \cdot C$, $B = c_{(m_0)} u^{\alpha} c_{(m_0)}$, $\alpha < n-2$, which, by the definition of the element $c_{(m_0)}$, are equal to zero. Thus $\mathfrak{N}^2 = 0$, which contradicts the condition $N(L) = 0$.

Lemma 2. *A Lie ring L of characteristic p, generated by generators x_i, $x_i^2 = 0$ ($i = 0, 1, \dots, d$; d arbitrary), and satisfying the n-th Engel condition, $n < p$, is nilpotent.*

First of all, let us note that one may restrict oneself to the particular case in which the additional condition $[x_i x_j] = 0$, $i, j = 1, 2, \dots, d$, is imposed on the generators of the ring L. This assertion follows from a more general theorem, whose proof (quite transparent) we omit.

According to Lemma 1, in L it suffices to find an element $c_{(2)}$. This can be obtained as follows.

1) If $c^2 = 0$, but $[ca^3c] \neq 0$ for some $a \in L$, then there is an element

$$e = e_1 + e_2 + e_3, \qquad e_i^2 = 0,$$

for which also $[ce^3c] \neq 0$.

Indeed, any element a of L can be written in the form $a = \sum a_i$, $a_i^2 = 0$, since from the conditions $h_1^2 = 0$ and $h_2^2 = 0$ it follows that $[h_1 h_2]^2 = 0$.

2) By direct verification one establishes that if $c^2 = cu^2cv^2c = 0$, then

$$[ce^3c]u^2[ce^3c] = 0.$$

Consequently, one may put $c_{(2)} = [ce^3c]$.

3) Let $c^2 = 0$ and $cx_ix_jc = 0$ for all $i, j > 0$. Then

$$T = c_0u^2c_0v^2c_0 = 0,$$

where

$$c_0 = [cb^3], \qquad b = x_\alpha + x_\beta + x_\gamma, \qquad \alpha, \beta, \gamma > 0.$$

Here it is first shown that the identities

$$T_1(u) = c_0u^2cb^3 = 0$$

and

$$T_2(u) = c_0u^2bcb^2 = 0$$

hold. From symmetry considerations one may likewise conclude that the expressions

$$\overline{T_1(u)} = b^3cu^2c_0$$

and

$$\overline{T_2(u)} = b^2cbu^2c_0$$

are equal to zero. After this the required relation is obtained:

$$T = \{T_1(u) - 3T_2(u)\}v^2c_0 - c_0u^2\{\overline{T_1(v)} - 3\overline{T_2(v)}\} = 0.$$

4) Define recursively the sequence of elements:

$$c_0 = [x_0 x_i], \quad i > 0, \qquad c_1 = [c_0 a_1^2 x_0], \dots, \qquad c_m = [c_{m-1} a_m^2 x_0],$$

$$a_k = x_{i_k} + x_{j_k}, \qquad i_k, j_k = 1, 2, \dots, d.$$

By induction on m the identities are verified: $c_m x_0 = x_0 c_m = 0$, $[c_m b^3 c_m] = 0$, $b = x_\alpha + x_\beta + x_\gamma,\ \alpha, \beta, \gamma > 0$.

Suppose now that $c_m \neq 0$, but $c_{m+1} = 0$, whatever elements a_k we choose in constructing the sequence $\{c_i\}$. Then the identity $c_m x_i x_j c_m = 0,\ i, j = 0, 1, \dots, d$, holds.

5) The finiteness of the number of nonzero terms in the sequence $\{c_i\}$ is proved as follows.
$c_{m+1} = [c_0 [x_0 a_1^2][x_0 a_2^2] \dots [x_0 a_{m+1}^2]] =$

$$= -[x_i x_0 y_1 y_2 \dots y_{m+1}].$$

The number of distinct $y_k = [x_0 a_k^2]$ does not exceed $\binom{d}{2}$, since $a_k = x_{i_k} + x_{j_k}$. By hypothesis, the Lie ring with the $(n-1)$-th Engel condition is locally nilpotent. Consequently, for sufficiently large m

$$y_1 y_2 \dots y_{m+1} = \sum u_i^{n-1} v_i^{r_i},$$

where u_i and v_i are elements of the Lie ring with generators $y_1, \dots, y_{\binom{d}{2}}$. Clearly, $[x_0 y_k] = 0$. Therefore

$$n x_0 u_i^{n-1} = \sum_{0 \le s \le n-1} u_i^s x_0 u_i^{n-1-s} = \sum z_j^n = 0,$$

i.e. $c_{m+1} = 0$, which was to be established.

The combination of assertions 1)–5) leads us to the conclusion that there exists in L an element $c_{(2)}$.

Lemma 3. For any pair of elements u, v of a Lie ring L with the n-th Engel condition and of characteristic $p > n + [n/2]$ (or $p = 0$), the identity

$$[uv^{n-1}]^2 = 0$$

holds.

Proof. Put

$$\alpha_i = (-1)^i \binom{n}{i},$$

$$\Delta_{\nu,\mu} = \begin{vmatrix} \alpha_\nu \dots \alpha_{\nu+\mu} \\ \dots \dots \dots \\ \alpha_{\nu+\mu} \dots \alpha_{\nu+2\mu} \end{vmatrix} = (-1)^{(\nu+\mu|2)(\mu+1)} \frac{\binom{n+\mu}{\mu+1}\binom{n+\mu-1}{\mu+1} \dots \binom{n-\nu+1}{\mu+1}}{\binom{\nu+2\mu}{\mu+1}\binom{\nu+2\mu-1}{\mu+1} \dots \binom{\mu+1}{\mu+1}},$$

$$\nu \geq 0, \qquad \mu \geq 0, \qquad \nu + \mu \leq n.$$

Obviously, $\Delta_{\nu,\mu} \neq 0$ if the characteristic $p > n + \mu$.

The assertion of the lemma follows from the equalities

$$A_{i,j} = v^j uv^{n-1-i} + uv^{n-1-i} = 0, \qquad 0 \leq j \leq i \leq n-1,$$

whose verification we shall now undertake. Since $v^{n-1}uv^{n-1} = 0$, we have $A_{0,j} = 0$. Suppose that the equality $A_{i,j} = 0$ has been proved for $i < k$ and all j, and we shall prove it for $i = k$. Further, we shall assume that $A_{k,j} = 0$ if $j < l$, i.e. it remains to establish this identity for $j = l, l+1, \dots, k,\ l \geq 0$. For $l = 0$ the condition $A_{k,j} = 0,\ j < 0$, is trivially fulfilled, so the induction is justified. Multiplying identity (2) on the left by $v^k uv^{n-1-k-s}$ and on the right by $v^s,\ s = 0, 1, \dots, n-1-k$, we obtain the system of identities

$$\sum_{0 \leq i \leq k-l} \alpha_{l+1+i+s} A_{k,l+i} = 0.$$

Consider separately two cases.

1) $n - k \geq k - l + 1$. Since $\Delta_{l+1,k-l} \neq 0$, from the first $k - l + 1$ equations of the system we find what is needed:

$$A_{k,l} = \dots = A_{k,k} = 0.$$

2) $n - k < k - l + 1$. Then express $A_{k,l}, \dots, A_{k,l+n-1-k}$ in terms of $A_{k,l+r}$, $r \geq n - k$, using the inequality $\Delta_{l+1,n-1-k} \neq 0$. Multiplying identity (2) on the left by v^s and on the right by $v^{l+r-s}uv^{n-1-l-r}$, $s = n-1-k, \dots, 1, 0$, we obtain

$$\sum_{0 \leq i \leq n-k-1} \alpha_{2k+l-n+i+s} A_{k+i,l+r} = 0.$$

From this system of $n - k$ equations with $n - k$ unknowns and determinant $\Delta_{2k+2-n,\,n-1-k} \neq 0$, we find $A_{k,\,l+r} = 0$ for any $r > n - k$. Therefore also $A_{k,\,l} = \dots = A_{k,\,l+n-1-k} = 0$. All determinants of the form $\Delta_{\nu,\mu}$ that occurred in the course of the proof were such that μ did not exceed the number $[n/2]$. The lemma is proved.

Theorem A is now obtained in an obvious way (see, for example, the proof of Theorem 6 in ([1])).

Let us note that the restriction on the characteristic arising in the proof of Lemma 3 can perhaps be reduced to $p > n + 1$ (for small n this has been checked), if one tries to use the identities (1) and (2) more fully. However, such an improvement would not be essential, since the most important case $p = n+1$ (connected with the weakened Burnside hypothesis on periodic groups for a prime exponent p) would still remain outside consideration. A possible way of solving this important problem consists in changing the method of proof of the existence of the element $c_{(2)}$.

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CITED LITERATURE

1. A. I. Kostrikin, *Izv. AN SSSR, ser. matem.*, **21**, no. 4 (1957).

Note: Figure translations are in progress. See original paper for figures.