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Abstract

Full Text

MATHEMATICS

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THREE-DIMENSIONAL SPACE WITH A CUBIC SEMIMETRIC

(Presented by Academician I. G. Petrovskii, 13 IX 1957)

Let $F_3^{(3)}$ be a three-dimensional Finsler space whose metric is given by the cubic differential form

$$ds^3 = a_{\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma \quad (\alpha, \beta, \gamma, \dots, \omega = 1, 2, 3)$$

with nonzero discriminant (⁽³⁾, p. 313). Recently K. Tonooka (⁽²⁾), extending to the case $F_3^{(3)}$ the method indicated by A. E. Liber in (⁽¹⁾), constructed a system of algebraic comitants for $F_3^{(3)}$ and used them to find a linear affine connection. In the present note, a connection in $F_3^{(3)}$ invariant with respect to a conformal transformation of the metric is constructed, and some questions of the differential geometry of the space X_3 with a given pseudotensor field whose discriminant is nonzero are also considered. We shall denote such a space by $\mathfrak{F}_3^{(3)}$.

1. Since the discriminant \mathfrak{A} of the tensor $a_{\alpha\beta\gamma}$ is nonzero, i.e. $\mathfrak{A} \neq 0$, the fundamental components $B^{\alpha\beta\gamma}$, $A^{\alpha\beta\gamma}$, $L_{\alpha\beta\gamma}$, $P_\mu^{\lambda\alpha\beta\gamma}$ of the fundamental tensor $a_{\alpha\beta\gamma}$ are successively determined (⁽²⁾) from the relations

$$a_{\alpha\beta\gamma} B^{\alpha\beta\gamma} = 0, \quad h_{\alpha\beta\gamma} B^{\alpha\beta\gamma} = \mathfrak{A}, \quad (1)$$

where $h_{\alpha\beta\gamma}$ are the coefficients of the Hessian of the cubic ternary form corresponding to the fundamental tensor;

$$A^{\alpha\beta\gamma} = \frac{1}{2} \varepsilon^{\alpha\alpha_1\alpha_2} \varepsilon^{\gamma\gamma_1\gamma_2} B^{\beta\beta_1\beta_2} a_{\alpha_1\beta_1\gamma_1} a_{\alpha_2\beta_2\gamma_2}, \quad (2)$$

by $\varepsilon^{\alpha\beta\gamma}$ is denoted the fundamental contravariant trivector density of weight +1;

$$A^{\alpha\beta\gamma} L_{\alpha\beta\gamma} = 0, \quad B^{\gamma\beta\gamma} L_{\alpha\beta\lambda} = \mathfrak{A} \delta_\lambda^\gamma; \quad (3)$$

$$P_{\mu}^{\lambda\alpha\beta\gamma} = \delta_{\mu}^{\lambda} A^{\alpha\beta\lambda} + \varepsilon^{\gamma\sigma_1\rho_1} \varepsilon^{\lambda\alpha\rho_2} a_{\sigma_1\sigma_2\mu} a_{\rho_1\rho_2\omega} B^{\alpha\beta\omega}. \quad (4)$$

The comitants listed are symmetric in the indices α, β, γ , except for $L_{\alpha\beta\gamma}$, which is symmetric only in the indices α, β , and they satisfy the relations

$$\text{a) } A^{\alpha\beta\gamma} a_{\alpha\beta\lambda} = \mathfrak{A} \delta_{\lambda}^{\gamma}, \quad \text{b) } B^{\gamma\beta\gamma} h_{\lambda\beta\lambda} = \frac{1}{3} \mathfrak{A} \delta_{\lambda}^{\gamma}; \quad (5)$$

$$\text{a) } P_{\mu}^{\lambda\alpha\beta\gamma} a_{\nu\beta\gamma} = \mathfrak{A} \delta_{\mu}^{\delta} \delta_{\nu}^{\lambda}, \quad \text{b) } P_{\mu}^{\lambda\alpha\beta\gamma} L_{\alpha\beta\gamma} = 0. \quad (6)$$

The linear connection in $F_3^{(3)}$ is completely determined ⁽²⁾ by the requirement

$$P_{\mu}^{\lambda\alpha\beta\gamma} \nabla_{\nu} a_{\alpha\beta\gamma} = 0,$$

and the connection coefficients $\Gamma_{\nu\mu}^{\lambda}$ have the form

$$\Gamma_{\nu\mu}^{\lambda} = \frac{1}{3\mathfrak{A}} P_{\mu}^{\lambda\alpha\beta\gamma} \partial_{\nu} a_{\alpha\beta\gamma}. \quad (7)$$

Theorem. For the mapping of $F_3^{(3)}$ into Minkowski space, it is necessary and sufficient that the torsion tensor of the connection and the vector density $\chi_{\nu} = B^{\alpha\beta\gamma} \nabla_{\nu} a_{\alpha\beta\gamma}$ vanish. (In ⁽²⁾, in the formulation of the theorem the requirement that the torsion tensor of the connection vanish was not taken into account.)

2. Let us now consider a conformal transformation of the metric in $F_3^{(3)}$, which is defined by the relation

$$*a_{\alpha\beta\gamma} = \sigma a_{\alpha\beta\gamma}. \quad (8)$$

Then the basic algebraic concomitants are transformed as follows:

$$*\mathfrak{A} = \sigma^{12} \mathfrak{A}, \quad *B^{\alpha\beta\gamma} = \sigma^9 B^{\alpha\beta\gamma}, \quad *A^{\alpha\beta\gamma} = \sigma^{11} A^{\alpha\beta\gamma}, \quad (9)$$

$$*P_{\mu}^{\lambda\alpha\beta\gamma} = \sigma^{11} P_{\mu}^{\lambda\alpha\beta\gamma}.$$

Let us establish a similar connection between the coefficients of the connection (7). Differentiating (8) with respect to ξ^{ν} , we have

$$\partial_{\nu} *a_{\alpha\beta\gamma} = \sigma_{\nu} a_{\alpha\beta\gamma} + \sigma \cdot \partial_{\nu} a_{\alpha\beta\gamma} \quad \left(\sigma_{\nu} = \frac{\partial_{\nu} \sigma}{\sigma} \right).$$

Then, by virtue of (6a), (8), and (9), we obtain

$$*P_{\mu}^{\lambda\alpha\beta\gamma}\partial_{\nu} *a_{\alpha\beta\gamma} = \sigma^{12} \left(P_{\mu}^{\lambda\alpha\beta\gamma}\partial_{\nu} a_{\alpha\beta\gamma} + \mathfrak{A}\delta_{\mu}^{\lambda}\sigma_{\nu} \right)$$

and, consequently,

$$*\Gamma_{\nu\mu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda} + \frac{1}{3}\sigma_{\nu}\delta_{\mu}^{\lambda}. \quad (10)$$

(10) shows that, under a conformal transformation of the metric, the object (7) is transformed projectively. Further, if $S_{\nu\mu}^{\lambda}$ is the torsion tensor of the connection (7), then under the transformation (8) $S_{\nu\mu}^{\lambda}$ is transformed as follows:

$$*S_{\nu\mu}^{\lambda} = S_{\nu\mu}^{\lambda} + \frac{1}{3}\sigma_{[\nu}\delta_{\mu]}^{\lambda}.$$

Contracting the last equality with respect to the indices λ, μ , then multiplying the result of the contraction by δ_{μ}^{λ} and subtracting what is obtained from (10), we have

$$*\Gamma_{\nu\mu}^{\lambda} = \overset{(k)}{\Gamma}_{\nu\mu}^{\lambda}, \quad \text{where}$$

$$\overset{(k)}{\Gamma}_{\nu\mu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda} - S_{\nu}\delta_{\mu}^{\lambda} \quad (S_{\nu} = S_{\nu\omega}^{\omega}). \quad (11)$$

The constructed connection coefficients (11) are invariant with respect to a conformal transformation of the metric and have the form

$$\overset{(k)}{\Gamma}_{\nu\mu}^{\lambda} = \frac{1}{3\mathfrak{A}} \left(P_{\mu}^{\lambda\alpha\beta\gamma}\partial_{\nu} a_{\alpha\beta\gamma} - P_{[\omega}^{\omega\lambda\beta\gamma}\partial_{\nu]} a_{\alpha\beta\gamma}\delta_{\mu}^{\lambda} \right).$$

From the results obtained, the following theorem easily follows:

Theorem 1. In order that $F_3^{(3)}$ be conformally flat, it is necessary and sufficient that the following conditions be fulfilled: the vanishing of the vector density $\chi_{\nu} = B^{\alpha\beta\gamma}\nabla_{\nu} a_{\alpha\beta\gamma}$, the semisymmetry ⁽⁴⁾ of the connection (7), and the vector S_{ν}^{ω} must be a gradient vector.

3. Let a space X_3 be given, in each local E_3 of which there is given a symmetric covariant pseudotensor $A_{\alpha\beta\gamma}$ of third valence, whose discriminant is distinct from zero. Since the discriminant of the pseudotensor $A_{\alpha\beta\gamma}$ is not equal to zero, we can construct a tensor density, determined up to sign, of weight -1 , with discriminant equal to ± 1 , in the following way:

$$\mathfrak{A}_{\alpha\beta\gamma} = \frac{A_{\alpha\beta\gamma}}{|\text{Dis}(A_{\alpha\beta\gamma})|^{1/12}} \quad (\tilde{\mathfrak{A}} = \text{Dis}(\mathfrak{A}_{\alpha\beta\gamma}) = \pm 1). \quad (12)$$

Now we can consider X_3 with a prescribed field of tensor density, determined up to sign, and whose discriminant satisfies condition (12). We shall denote such a space by $\mathfrak{F}_3^{(3)}$, and we shall call the prescribed tensor density the **fundamental tensor density**.

Substituting into relations (1)–(4), instead of the tensor $a_{\alpha\beta\gamma}$, the fundamental tensor density $\mathfrak{A}_{\alpha\beta\gamma}$, we construct the tensor densities $\tilde{B}^{\alpha\beta\gamma}$, $\tilde{A}^{\lambda\alpha\beta\gamma}$, $\tilde{L}_{\alpha\beta\gamma}$, $\tilde{P}_\mu^{\lambda\alpha\beta}$ of weights $+1, +1, -1, +1$, respectively, which are the fundamental algebraic concomitants of the space $\mathfrak{F}_3^{(3)}$. These concomitants satisfy relations analogous to (5), (6). A linear symmetric connection in $\mathfrak{F}_3^{(3)}$ can now be defined by the conditions

$$\text{a) } \tilde{P}_{(\mu}^{\lambda\alpha\beta} \nabla_{\nu)} \mathfrak{A}_{\alpha\beta\gamma} = 0, \quad \text{b) } S_{\nu\mu}^\lambda = 0. \quad (13)$$

Expanding the covariant differentiation in (13a) and using a relation analogous to (6a) for the tensor densities $\mathfrak{A}_{\alpha\beta\gamma}$, $\tilde{P}_\mu^{\lambda\alpha\beta}$, we have

$$\tilde{P}_{(\mu}^{\lambda\alpha\beta} \partial_{\nu)} \mathfrak{A}_{\alpha\beta\gamma} - 3\mathfrak{A} \Gamma_{\nu\mu}^\lambda + \mathfrak{A} \Gamma_{(\nu} \delta_{\mu)}^\lambda = 0.$$

Contracting with respect to the indices λ, μ , in view of (13b), we obtain

$$\Gamma_\mu^\lambda = \frac{1}{\mathfrak{A}} \tilde{P}_{(\mu}^{\omega\alpha\beta\gamma} \partial_{\omega)} \mathfrak{A}_{\alpha\beta\gamma},$$

and, consequently,

$$\Gamma_{\nu\mu}^\lambda = \frac{1}{3\mathfrak{A}} \left(\tilde{P}_\mu^{\lambda\alpha\beta\gamma} \partial_\nu \mathfrak{A}_{\alpha\beta\gamma} + \tilde{P}_{(\nu}^{\omega\alpha\beta\gamma} \delta_{\mu)}^\lambda \mathfrak{A}_{\alpha\beta\gamma} \right). \quad (14)$$

Putting

$$\varphi_{\nu\mu}^\lambda = \frac{1}{\mathfrak{A}} \tilde{P}_\mu^{\lambda\alpha\beta\gamma} \nabla_\nu \mathfrak{A}_{\alpha\beta\gamma}, \quad \psi_\nu = \frac{1}{3\mathfrak{A}} \tilde{B}^{\alpha\beta\gamma} \nabla_\nu \mathfrak{A}_{\alpha\beta\gamma},$$

it is not difficult to show that, for the covariant derivative of the fundamental tensor density $\mathfrak{A}_{\alpha\beta\gamma}$, the following equality holds:

$$\nabla_\nu \mathfrak{A}_{\alpha\beta\gamma} = \varphi_{\nu(\alpha}^\omega \mathfrak{A}_{\beta\gamma)\omega} + \psi_\nu \tilde{L}_{(\alpha\beta\gamma)}.$$

The tensor $\varphi_{\nu\mu}^\lambda$, as is easy to see, is skew-symmetric in the indices ν, μ .

Let us apply the Ricci identity to the fundamental tensor density $\mathfrak{A}_{\alpha\beta\gamma}$. We have

$$2\nabla_{[\lambda}\nabla_{\nu]}\mathfrak{A}_{\alpha\beta\gamma} = -3R_{\lambda\nu(\alpha}^{\omega}\mathfrak{A}_{\beta\gamma)\omega} + R_{\lambda\nu\omega}^{\omega}\mathfrak{A}_{\alpha\beta\gamma}.$$

Let now $\varphi_{\nu\mu}^{\lambda} = 0$, $\psi_{\nu} = 0$; then, multiplying the Ricci identity by $\tilde{P}_{\mu}^{\tau\alpha\beta\gamma}$, we obtain

$$3R_{\lambda\nu\mu}^{\tau} - R_{\lambda\nu\omega}^{\omega}\delta_{\mu}^{\tau} = 0. \quad (15)$$

It is well known ⁽⁴⁾ that in a space of symmetric affine connection the curvature tensor satisfies the identity $R_{[\lambda\nu\mu]}^{\tau} = 0$; then from (15) we have $R_{[\lambda\nu|\omega]}^{\omega}\delta_{\mu}^{\tau} = 0$, which gives $R_{\lambda\nu\omega}^{\omega} = 0$ and, consequently, $R_{\lambda\nu\mu}^{\tau} = 0$.

Theorem 2. *A necessary and sufficient condition for there to exist in $\mathfrak{F}_3^{(3)}$ a coordinate system in which the components of the fundamental tensor density $\mathfrak{A}_{\alpha\beta\gamma}$ do not depend on the coordinates of a point of $\mathfrak{F}_3^{(3)}$, is the vanishing of the tensor $\varphi_{\nu\mu}^{\lambda}$ and the vector ψ_{ν} .*

Proof. Let $\partial_{\nu}\mathfrak{A}_{\alpha\beta\gamma} = 0$; then, as is seen from (14), $\Gamma_{\nu\mu}^{\lambda} = 0$, and, consequently, $\varphi_{\nu\mu}^{\lambda} = 0$, $\psi_{\nu} = 0$. Conversely, let $\varphi_{\nu\mu}^{\lambda} = 0$, $\psi_{\nu} = 0$; then $R_{\lambda\nu\mu}^{\tau} = 0$, consequently, there exists a coordinate system in which $\Gamma_{\nu\mu}^{\lambda} = 0$, and hence in this coordinate system $\partial_{\nu}\mathfrak{A}_{\alpha\beta\gamma} = 0$.

It is also possible to prove the following theorem.

Theorem 3. Every differential concomitant of the fundamental tensor density $\mathfrak{A}_{\alpha\beta\gamma}$ is an algebraic concomitant of $\mathfrak{A}_{\alpha\beta\gamma}$, $\varphi_{\mu\nu}^{\lambda}$, ψ_{ν} , and their covariant derivatives computed with respect to the object (14).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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