MATHEMATICS

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Abstract

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ON THE APPROXIMATE CONSTRUCTION OF CERTAIN CONFORMAL MAPPINGS

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It is known that the problem of conformally mapping the circle $|z| < 1$ onto a domain of the w-plane containing the point $w_0 = 0$ and bounded by a simple, closed contour L, star-shaped with respect to the point w_0, given by the equation

$$w = \exp\{f(t) + it\}, \qquad f(t + 2\pi) = f(t), \tag{1}$$

under the condition that

$$w(0) = 0, \qquad w'(0) > 0, \tag{2}$$

reduces to a nonlinear singular integral equation for the function $t = t(\varphi)$, $t(\varphi + 2\pi) = t(\varphi) + 2\pi$:

$$t(\varphi) - \varphi = Sf[t(\varphi)]; \tag{3}$$

here and in what follows S denotes the integral operator with Hilbert kernel

$$Sh(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} h(\sigma) \operatorname{ctg} \frac{\varphi - \sigma}{2} \, d\sigma. \tag{4}$$

Equation (3) admits the application of iterations

$$t_{n+1}(\varphi) = \varphi + Sf[t_n(\varphi)]$$

(Theodorsen' s method). The convergence of the process was proved in the work ([1]) under the additional assumption that

$$|f'(t)| \le q < 1. \tag{5}$$

In the present note we consider the case of a non-star-shaped contour L, defined by the equation

$$w = \exp\{f_1(t) + i f_2(t)\}, \tag{6}$$

where

$$f_1(t + 2\pi) = f_1(t), \qquad f_2(t + 2\pi) = f_2(t) + 2\pi,$$

$$(f_1')^2 + (f_2')^2 \geq d^2 > 0.$$

It is assumed that f_1 and f_2 are twice continuously differentiable functions, and moreover their second derivatives satisfy the Lipschitz condition

$$|f_k''(t_1) - f_k''(t_2)| \leq l_k |t_1 - t_2|, \qquad k = 1, 2. \tag{7}$$

The problem analogous to the preceding one, with condition (2), can be reduced to the nonlinear singular integral equation

$$P[t(\varphi)] \equiv f_2[t(\varphi)] - \varphi - S f_1[t(\varphi)] = 0. \tag{8}$$

Theodorsen' s method is inapplicable here, since for an unknown contour the function f_2 is not monotone. It turns out to be possible to apply, for solving this equation (and also in the preceding case—without restriction (5)), Newton' s method, developed by L. V. Kantorovich for the approximate solution of nonlinear functional equations ([2,3]). With this method the successive approximations are determined from linear singular integral equations (we give them for the modified method)

$$f_2'[t_0](t_{n+1} - t_n) - S f_1'[t_0](t_{n+1} - t_n) = -P[t_n]. \tag{9}$$

Here $n = 0, 1, 2, \ldots$. Solving this equation explicitly, we arrive at algorithm (14) (see below). To solve equation (9), whose index ([4]) turns out to be equal to zero, we transform it into a Hilbert problem, somewhat modifying the technique for solving the equation conjugate with the characteristic one (([4]), p. 133).

Let H_μ be the space of 2π-periodic functions $u(\varphi)$ satisfying the Hölder condition, and let the norm in it be defined in the usual way by $\|u\| = \max|u| + A_u$, where A_u is the Hölder constant. Introduce the notation:

$$f_2'(t) - i f_1'(t) = g(t) e^{i\omega(t)}, \qquad g(t) \geq d > 0,$$

$$\frac{\cos\omega[t_0(\varphi)]}{g[t_0(\varphi)]} = h_1(\varphi), \qquad \frac{\exp\{-S\omega[t_0(\varphi)]\}}{g[t_0(\varphi)]} = h_2(\varphi),$$

$$\exp\{+S\omega[t_0(\varphi)]\}\sin\omega[t_0(\varphi)] = h_3(\varphi),$$

$$\|P[t_0(\varphi)]\| = \bar{\eta}_0; \qquad \frac{1}{2\pi}\int_0^{2\pi}\omega[t_0(\varphi)]\,d\varphi = \omega_c, \tag{10}$$

$$B_0 = \|h_1(\varphi)\| + \|h_2(\varphi)\|\|h_3(\varphi)\|(N_0 + \operatorname{tg}|\omega_c|),$$

$$K = (l_1 + N_0 l_2)(2\pi + \|t_0(\varphi) - \varphi\| + 2B_0\bar{\eta}_0).$$

Here N_0 is an estimate for the norm of the linear operator S (see (4)), which, by a known theorem of Privalov, maps H_μ into itself; for example, one can show that

$$\|S\| \le N_0 = \max\left[\frac{1}{\mu\pi}(1 + 2^{1+\mu} + 3^\mu) + \frac{\pi}{1-\mu}; \frac{4^{1+\mu}}{\mu\pi}\right] + \frac{2\pi^\mu - 1}{\mu}.$$

Applying theorem 1 of the paper ([5]) (for $\alpha = 1$), we arrive at the following result.

Theorem. Suppose that, for equation (8) and the initial approximation $t_0(\varphi)$, the conditions

$$\cos\omega_c \ne 0, \tag{11}$$

$$h_0 = B_0^2 K\bar{\eta}_0 < \tfrac{1}{2}. \tag{12}$$

are satisfied. Then in the domain

$$\|t(\varphi) - t_0(\varphi)\| \le 2B_0\bar{\eta}_0 \tag{13}$$

there exists a solution $t^*(\varphi)$ of equation (8), and the successive approximations $\{t_n(\varphi)\}$:

$$t_{n+1}(\varphi) = t_n(\varphi) - h_1(\varphi)P[t_n(\varphi)] +$$

$$+h_2(\varphi)\left\{S(h_3(\varphi)P[t_n(\varphi)]) - \frac{\operatorname{tg}|\omega_c|}{2\pi}\int_0^{2\pi} h_3(\varphi)P[t_n(\varphi)]\,d\varphi\right\} \tag{14}$$

converge to $t^*(\varphi)$:

$$\|t_n(\varphi) - t^*(\varphi)\| \leq \frac{q^n}{1-q} B_0 \overline{\eta}_0, \qquad q = 1 - \sqrt{1 - 2h_0} < 1. \tag{15}$$

Remark. Condition (11) is always satisfied if the contour L is star-shaped or if it has an axis of symmetry (in the latter case $\omega_c = 0$). In the general case one can prove that condition (11) will be satisfied if

$$|t_0(\varphi) - t_i^*(\varphi)| \leq \frac{\pi}{2 \max |\omega'(t)|}.$$

Condition (12) is also satisfied in some neighborhood of the solution $t^*(\varphi)$, whose existence in the space under consideration is evident. We also note that the functions $t(\varphi) \in H_\mu$; however, the substitution $t(\varphi) - \varphi = u(\varphi)$ leads to elements $u(\varphi) \in H_\mu$. The functions $t(\varphi)$ have been retained in the formulas for the sake of brevity; finally, $t(\varphi) - t_0(\varphi) \in H_\mu$.

Other results on Newton' s method are applied in a similar way, for example, the convergence theorems for the basic method.

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Note: Figure translations are in progress. See original paper for figures.