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Abstract

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MATHEMATICS

P. V. NIKOLAEV

ON THE UNIQUENESS OF NOMOGRAPHIC REPRESENTATIONS OF EQUATIONS

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The problem of the uniqueness of the representation by net nomograms of an equation in three variables has been considered by many authors ([1–5]), etc., but has not yet been solved. Let us consider it for an equation, analytic in some domain G, in three pairs of variables

$$F(t, \tau) = F(t_1, \tau_1;\ t_2, \tau_2;\ t_3, \tau_3) = 0, \tag{1}$$

admitting in G an anamorphosis

$$\Phi(t, \tau) = \psi(t, \tau) \cdot F(t, \tau) = |f_{i1}(t_i, \tau_i);\ f_{i2}(t_i, \tau_i);\ f_{i3}(t_i, \tau_i)| \tag{2}$$

with a prescribed partition of the variables into pairs. Here $\Phi(t, \tau)$ is also a function analytic in G ($\Phi \not\equiv 0$);

$$\psi(t, \tau) = \psi_1(t_2, \tau_2;\ t_3, \tau_3) \cdot \psi_2(t_3, \tau_3;\ t_1, \tau_1) \cdot \psi_3(t_1, \tau_1;\ t_2, \tau_2); \tag{3}$$

A is a multiplier of equation (1). We shall assume that F is a nondegenerate function, i.e., different from a function of the form (3). The equations $F = 0$ and $\Phi = 0$ will be called *similar*, in particular, N-equivalent, if ψ has the form $\varphi_1(t_1, \tau_1) \cdot \varphi_2(t_2, \tau_2) \cdot \varphi_3(t_3, \tau_3)$.

In the case of an algebraic equation in three variables

$$F(t) = F(t_1, t_2, t_3) = 0 \tag{4}$$

the problem of anamorphosis, as follows from the lemmas on the rationality of anamorphoses ([6]), is also solved in the class of algebraic equations; moreover, the bases C_i and $\overline{C}_i$ of the variable t_i in any two anamorphoses of equation (4) will be birationally equivalent. If, therefore, in an anamorphosis (1) (with

$\tau_i = \text{const}$) of equation (4) birational parameters have been introduced for the curves C_i,

$$s_i = f_i(t_i) \quad (i = 1, 2, 3), \tag{5}$$

i.e., Lüroth parameters ([7]), then the s_i will be Lüroth parameters in every other anamorphosis of equation (4) as well. Thus the problem is reduced, by the change of variables (5) from t_i to s_i, to the consideration, instead of (4), of a birational equation in the variables s_i similar to it, i.e., an equation in whose anamorphoses the points of the base C_i are birational functions of the parameter s_i.

Lemma 1 (on the uniqueness of anamorphoses). *Suppose that in each of two nomograms of some nondegenerate equation $F(t) = 0$, analytic in the domain G, there exists the following configuration of points: three noncollinear points A_k $(k = 1, 2, 3)$ with labels, respectively, $t_3 = t_{3k}$*

$(k = 1, 2, 3)$, the line A_1A_3 with the points $t_i = t_{i1}$ $(i = 1, 2)$, and the line A_2A_3 with the points $t_i = t_{i2}$ $(i = 1, 2)$. If the points of the nomogram $t_i = t_{i1}$ $(i = 1, 2)$, distinct from one another, and also the points $t_i = t_{i2}$ $(i = 1, 2)$ (possibly coincident) are different from the vertices of the triangle $A_1A_2A_3$, then these nomograms are projective.

Indeed, let (2) (for $\tau_i = \text{const}$) and

$$\bar{\Phi}|t| = \bar{\psi}(t) \cdot \Phi(t) = |\bar{f}_{i1}(t_i);\ \bar{f}_{i2}(t_i);\ \bar{f}_{i3}(t_i)| \tag{6}$$

be two such anamorphoses. Taking $A_1A_2A_3$ as the coordinate triangle in (2) and (6), one may, with a proper choice of the unit point, write the expansion of the function $\Phi(t)$ with respect to t_3 in the form

$$\bar{\Phi}(t) = \bar{\psi}_3(t_1, t_2) \cdot \sum_{k=1}^{3} \bar{\psi}_1^{(0k)} \bar{\psi}_2^{(k0)} \Phi^{(00k)} \bar{f}_{3k}, \tag{7}$$

where $\Phi^{(00k)} = \Phi(t_1, t_2, t_{3k})$, $\bar{\psi}_1^{(0k)} = \bar{\psi}_1(t_2, t_{3k})$, etc.

Applying to (7) the condition of an anamorphosis with a simple A-multiplier ([8]), we obtain the identities

$$\sum_{k=1}^{3} \bar{f}_{ik}(t_i) \cdot \bar{\psi}_1^{(0k)} \bar{\psi}_2^{(0k)} \Phi^{(00k)} \equiv 0 \quad (i = 1, 2). \tag{8}$$

It can be shown that from (8), under the conditions of the lemma, the elements $\bar{f}_{ik}$ $(i = 1, 2;\ k = 1, 2, 3)$ are determined, in essence, uniquely; but then this will also be true for the elements $\bar{f}_{3k}$ $(k = 1, 2, 3)$. Hence it follows:

Theorem 1. All anamorphoses of an algebraic (irreducible) equation (4) which is not similar to an equation of the third N-order are projective.

Indeed, equation (4) may be regarded as a reduced equation, i.e., one free of divisors with two variables. If it admits only a simple A-multiplier, then the question is clear ([8]). If, however, it admits a general A-multiplier, then the order of the basis common to all variables will be greater than 3, and therefore any two nomograms of such an equation will have the configuration of points of Lemma 1; the theorem follows from this.

In the case where equation (1) is pseudoalgebraic ([9]) in three variables,

$$F(t) = \sum_{i=0}^{m} a_i(t_2, t_3) \cdot t_1^i = 0, \tag{9}$$

where the a_i are functions analytic in G, it proves possible to consider it in a certain cylindrical domain H

$$|t_1| < \infty; \qquad |t_i - \alpha_i| < h \quad (i = 1, 2, 3), \tag{10}$$

where $t_i = \alpha_i$ $(i = 1, 2, 3)$ is the point $\alpha \in G$. Namely, it is not difficult to show that similarity relations and N-equivalence of equations holding locally (in some subdomain $E \subset G$) will also hold "as a whole" (in the entire domain G). But from this, by virtue of the lemmas on the rationality of anamorphoses ([6]), it will follow that an anamorphosis constructed in $E \subset G$ for the equation $F(t) = 0$ will, up to N-equivalence, represent it also in G, or in H, having a nonempty intersection with G. It is also possible to show that, in a linear anamorphosis of equation (9) given in the domain H, the basis C_2 will then and only then lie on C_1 if the pseudopolynomial $F(t)$ (with vertex at (α_2, α_3)) is algebraically reducible ([9]) and has a nontrivial algebraic divisor in t_1 and t_2. Hence it follows that, for equation (9), freed of algebraic divisors and admitting an A-multiplier with the variable t_1, a change of variables of the form (5) is possible

algebraization, at least with respect to one of the variables t_2, t_3. Therefore the following theorem is valid.

Theorem 2. *All anamorphoses of a nondegenerate pseudoalgebraic equation (9) that is not similar to an equation of the third N-order are projective.*

In the case of equation (1) in three pairs of variables, the following is valid:

Theorem 3. *If a nondegenerate analytic equation (1), algebraic (or algebraizable) with respect to at least one variable, is not similar to an equation of the third N-order, then all its anamorphoses (2) with a prescribed partition of the variables into pairs (t_i, τ_i) are projective.*

If one were to assume that such an equation has nonprojective anamorphoses, then it would be possible, by substitutions of the form $\tau_i = f_i(t_i)$ $(i = 1, 2, 3)$, where f_i are polynomials, to construct a pseudoalgebraic equation $F(t) = 0$ which, contrary to Theorem 2, would admit nonprojective anamorphoses.

In the general case of a nonalgebraizable equation (1), suppose that equation (1) has a direct anamorphosis

$$F(t,\tau) = |f_{i1}(t_i,\tau_i);\ f_{i2}(t_i,\tau_i);\ f_{i3}(t_i,\tau_i)| \tag{11}$$

and a nonprojective anamorphosis to it

$$\Phi(t,\tau) = \psi(t,\tau)\cdot F(t,\tau) = \left|\bar{f}_{i1}(t_i,\tau_i);\ \bar{f}_{i2}(t_i,\tau_i);\ \bar{f}_{i3}(t_i,\tau_i)\right|. \tag{12}$$

It can be shown that the dimensionality of any of the equations

$$\Phi_i(t,\tau) = \Phi(t,\tau) : \psi_i = 0 \qquad (i = 1,2,3) \tag{13}$$

with respect to each of the pairs of variables (t_k, τ_k) is equal to three, and therefore in the linear expansion

$$\Phi_3(t,\tau) = \sum_{k=1}^{3} \psi_1^{(0k)} \psi_2^{(k0)} F^{(00k)} \bar{f}_{3k} \tag{14}$$

among the system of generators, with respect to (t_1, τ_1), of this function:

$$\varphi_{11} = \psi_2^{(10)} f_{12}, \dots;\ \varphi_{16} = \psi_2^{(30)} f_{11}; \dots \tag{15}$$

three functions are linear combinations of the others; an analogous conclusion holds for the functions $\varphi_{2k} = \varphi_{2k}(t_2, \tau_2)$.

Considering further the two possible cases: linear independence or dependence of the functions φ_{2k} $(k = 1,2,3)$, we find that in each of them the anamorphosis condition for equation (1) leads to a relation of the third degree with respect to the functions f_{1k} $(k = 1,2,3)$. Thus, in the first case, the A-matrix $T_3^{(23)}$ of equation (14) has the form

$T_3^{(23)}$	φ_{21}	φ_{22}	φ_{23}
$\bar{f}_{31}$	$-\varphi_{12}$	φ_{11}	0
$\bar{f}_{32}$	$-\gamma_1\varphi_{13}$	$-\gamma_2\varphi_{13}$	$-\varphi_{14} - \gamma_3\varphi_{13}$
$\bar{f}_{33}$	$\alpha_1\varphi_{16} - \beta_1\varphi_{15}$	$\alpha_2\varphi_{16} - \beta_2\varphi_{15}$	$\alpha_3\varphi_{16} - \beta_3\varphi_{15}$

(16)

where α, β, γ are certain constants, and the anamorphosis condition $\left|T_3^{(23)}\right| = 0$ [8] leads to the aforementioned relation between the f_{1k}. If this relation is an equation for f_{1k} (and not an absolute identity), then it is clear that the variables (t_1, τ_1) have a binary scale on a curve of the third order of genus one. If, however, this relation is an absolute identity,

then, for the functions φ_{1k}, it turns out that the relation

$$\varphi_{11}\varphi_{14}\varphi_{16} - \varphi_{12}\varphi_{13}\varphi_{15} = 0, \tag{17}$$

will hold, from which the same conclusion will follow for the binary scale of the variables (t_1, τ_1); an analogous result will hold for each of the other pairs of variables.

The application of Lemma 1 leads to the following theorem.

Theorem 4. *If a nonalgebraizable equation (1) admits nonprojective anamorphoses (2), then each of the binary fields degenerates into a binary scale on an elliptic curve of the third order, common to all three pairs of variables.*

Consequently, the following theorem is valid:

General theorem. *If an analytic nondegenerate equation (1) admits nonprojective anamorphoses of the form (2), then all the binary fields (t_i, τ_i) degenerate into binary scales on algebraic curves; the sum of the orders of the distinct curves in each anamorphosis is equal to three. If such an equation is algebraizable with respect to at least one of the variables, then it is similar to an equation of the third N-order and, consequently, admits anamorphoses of all three genera; in its nomograms of the third genus with a common base there serves a universal curve of the third order. If, however, equation (1) is not algebraizable, then it admits only nomograms of the third genus with a common base on an elliptic curve of the third order.*

Remark. Using the parametrization of an elliptic curve by Weierstrass functions and Gronwall's work [2], we conclude that equation (1) admits nonprojective anamorphoses if and only if it is either similar to an equation of the third N-order or similar to an equation equivalent to it.

Ural Polytechnic Institute
named after S. M. Kirov

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Note: Figure translations are in progress. See original paper for figures.