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Abstract

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ON INDUCTIVE LIMITS OF NORMED SPACES

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In the works of Sebastião e Silva (1) and D. A. Raikov (2), inductive and projective limits of normed spaces with completely continuous embeddings are considered. The present note is devoted mainly to inductive and projective limits of reflexive normed spaces. The previously obtained results (1,2) carry over, in the main, to the case under consideration.

Definition 1. Let E_n be a sequence of locally convex spaces, where E_n is a linear subspace of E_{n+1} and the embedding (identification operation) of E_n into E_{n+1} is continuous. The **inductive limit** of the sequence of spaces E_n is their union

$$E = \bigcup_{1}^{\infty} E_n,$$

endowed with the strongest of the locally convex topologies for which all the embeddings of E_n into E are continuous.

Definition 2. Let E_n be the inductive limit of the spaces E_n, and suppose there exists a sequence of absolutely convex * closed neighborhoods of zero V_n (V_n is a neighborhood of zero in E_n) such that, for some $\lambda_n > 0$, $\lambda_n V_n \subset V_{n+1}$, $n = 1, 2, \ldots$. We shall say that the sequence of spaces E_n satisfies **condition** (F_1) if all V_n are closed in E, while V_1 contains no nonzero linear subspace of E; and, respectively, **condition** (F_2), if every absolutely convex closed neighborhood of zero W_n in E_n that is contained in V_n is closed in E_{n+1}.

Theorems 1 and 2 are proved under the assumption that one of the conditions (F_1) or (F_2) is satisfied.

Theorem 1. *Let E be the inductive limit of separable spaces E_n. Then E is separable, and every set bounded in E is contained in some E_n.*

Theorem 2. *Let E be the inductive limit of a sequence of normed spaces E_n. Then every set bounded in E is contained and bounded in some E_n.*

A locally convex space E is called **bornological** if every convex set that absorbs every bounded set is a neighborhood of zero in E.

Theorem 3. *In order that a separable bornological space have a metrizable conjugate, it is necessary and sufficient that it be the inductive limit of a sequence of normed spaces satisfying condition* (F_1).

* A subset A of a linear set L is called absolutely convex if, for any $x, y \in A$ and $|\alpha| + |\beta| \leq 1$, one has $\alpha x + \beta y \in A$.

Since the inductive limit of bornological (in particular normed) spaces is again a bornological space, from Theorems 2 and 3 we obtain:

Corollary. If E is the inductive limit of a sequence of normed spaces satisfying condition (F_2), then E is also the inductive limit of a sequence of normed spaces satisfying condition (F_1).

With the help of Grothendieck' s results ([3]), we obtain that every reflexive space with a metrizable dual is the inductive limit of a sequence of normed spaces satisfying condition (F_1).

Definition 3. Let E_n be a sequence of locally convex spaces, and let π_m^n, for all m and $n > m$, be continuous linear mappings of E_n into E_m such that $\pi_m^p = \pi_m^n \pi_n^p$ for $m < n < p$. Let E be the set of sequences $x = \{x_n\}$, where each $x_n \in E_n$ and $x_m = \pi_m^n(x_n)$ for all m and $n > m$, and let π_n be the linear mapping (projection) of E into E_n $(\pi_n(x) = x_n)$. The **projective limit** of the sequence of spaces E_n with respect to the mappings π_m^n is the space E, endowed with the weakest topology for which all projections π_n are continuous.

We shall call inductive and projective limits of reflexive normed spaces, respectively, spaces of classes (J) and (P).

It is clear that in this case both conditions (F_1) and (F_2) are satisfied.

Theorem 4. *Spaces of classes* (J) *and* (P) *are reflexive, and their strong duals are, respectively, spaces of classes* (P) *and* (J).

We agree to denote the weak topologies in a locally convex space E and in the dual space E', respectively, by $\sigma(E, E')$ and $\sigma(E', E)$.

Theorem 5. *Let* E *be the inductive limit of reflexive normed spaces* E_n, $A \subset E$, *and let* $A_n = A \cap E_n$ *be closed in the topology* $\sigma(E_n, E_n')$. *Then* A *is closed in* E *(in the strong topology).*

Remark. If A is a convex set, then from the closedness of A_n in E_n (in the strong topology) it follows that A is closed in E in the topology $\sigma(E, E')$.

The last remark somewhat strengthens a known result ([6]) (which, however, concerns a broader class of spaces) stating that in a space F' dual to a complete locally convex metrizable space F, for a convex set $A \subset F'$ to be closed in the topology $\rho(F', F)$, it is sufficient that all intersections of A with absolutely convex weakly closed bounded sets be weakly closed.

For quotient spaces and subspaces of classes (P) and (J), the duality established by D. A. Raikov ([2]) in the case of inductive and projective limits with completely continuous embeddings is preserved to a considerable extent.

Theorem 6. *Let F be a space of class (P), and let M be a closed subspace of F. The quotient space F/M is again a space of class (P).*

Theorem 7. *Let E be a space of class (J), and let N be a closed subspace of E. The quotient space E/N is again a space of class (J).*

Theorem 8. *A closed subspace M of a space F of class (P) is again a space of class (P), isomorphic to the strong dual of the space F'/M^0.*

We shall say that a linear mapping of one locally convex space into another is **weakly completely continuous** if the closure of the image of some neighborhood is weakly bicompact.

Let (J_1) be the class of inductive limits of normed spaces with weakly continuous embeddings, and let (J_1') be the class of conjugate spaces.

The spaces of class (J_1) are reflexive, and for spaces (J_1') and (J_1) Theorems 2 and 5–8 hold.

It is easy to prove that the inductive limit of arbitrary separable locally convex spaces with weakly completely continuous embeddings is a space (J_1).

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Note: Figure translations are in progress. See original paper for figures.