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M. M. POSTNIKOV

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MATHEMATICS

M. M. POSTNIKOV

LIMITING COMPLEXES OF CUBIC RESOLVENTS

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The main purpose of the present note is to give an internal characterization of cubic complexes that are limiting complexes of cubic resolvents (^{1}). As an application of the resulting theorem, cubic resolvents of arbitrary topological spaces are constructed. In the last section some simple remarks on resolvents of loop spaces are given.

1. ***EN*-complexes.** Let K be an arbitrary cubic complex. Following Kan (^{2}), we shall call an (n, ε, i)-**equation** in K a system of $(n-1)$-dimensional cubes $\sigma^j_\omega \in K$, where $\omega = 0, 1;\ 1 \leq j \leq n$, with $j \neq i$ if $\omega = \varepsilon$, satisfying the relations

$$\sigma^j_\omega \xi^{k-1} = \sigma^k_\zeta \omega^j,$$

where $\omega, \zeta = 0, 1$ and $j < k$, with $j \neq i$ if $\omega = \varepsilon$, and $k \neq i$ if $\zeta = \varepsilon$. We shall call the equation $\{\sigma^j_\omega\}$ **solvable** if there exists an n-dimensional cube σ such that $\sigma\omega^j = \sigma^j_\omega$. In this case the cube $\sigma\varepsilon^i$ is called a **solution** of the given (n, ε, i)-equation. A complex K is called an E-**complex** if every equation in K (for arbitrary n, ε, and i) is solvable. Two n-dimensional cubes $\sigma_0, \sigma_1 \in K$ are called **comparable** if $\sigma_0\varepsilon^i = \sigma_1\varepsilon^i$ for any $\varepsilon = 0, 1$ and any $i = 1, \dots, n$. Comparable cubes σ_0, σ_1 are called **homotopic** if there exists an $(n+1)$-dimensional cube σ such that $\sigma\varepsilon^1 = \sigma_\varepsilon$ and $\sigma\varepsilon^i = \sigma_0\eta^1\varepsilon^i$ for $i > 1$. An E-complex K will be called an EN-**complex** if: 1) it contains only one zero-dimensional cube, and 2) any two comparable but noncoinciding cubes of the complex K are not homotopic.

Theorem. *A cubic complex K is isomorphic to a limiting complex of some cubic resolvent if and only if it is an EN-complex.*

We outline the proof of the sufficiency of this condition (the proof of necessity is considerably simpler, and we omit it). Let K be an arbitrary EN-complex. A cube $\sigma \in N$ of dimension $n > 0$ will be called **belonging** to an $(n-1)$-dimensional cube ψ if $\sigma\varepsilon^1 = \psi$ and $\sigma\varepsilon^i = \psi\eta^1\varepsilon^i$, where $\varepsilon = 0, 1$, and $i > 1$. Let $\pi_n(K, \psi)$ be the totality of all n-dimensional cubes of the complex K belonging to the given $(n-1)$-dimensional cube ψ. Following Kan (2), we shall call the **sum** (in the case $n = 1$, the **product**) of cubes $\sigma, \tau \in \pi_n(K;\psi)$ the solution (as is easily seen, unique) of the $(n+1,1,1)$-equation $\{\sigma_\omega^j\}$, where

$$\sigma_0^1 = \sigma, \quad \sigma_1^2 = \tau$$

and $\sigma_\omega^j = \sigma\eta^1\omega^j$ for the remaining admissible values of ω and j. It turns out that, with respect to the operation thus defined, the set $\pi_n(K;\psi)$ is a group (for $n > 1$, abelian). The neutral element of this group is the cube $\psi\eta^1$. It is easy to see that for any $(n-1)$-dimensional cubes ψ_0 and ψ_1 and any one-dimensional cube γ there exists (at least one) n-dimensional cube $\bar{\gamma}$ for which $\bar{\gamma}\varepsilon^1 = \psi_\varepsilon$ and $\bar{\gamma}\bar{0}^2 \dots 0^2 = \gamma$. It turns out that for any cube $\sigma \in \pi_n(K;\psi_1)$ the solution τ of the $(n+1,0,2)$-equation $\{\sigma_\omega^j\}$, where $\sigma_0^1 = \bar{\gamma}$, $\sigma_1^1 = \sigma$, and $\sigma_\omega^j = \gamma\eta^1\omega^j$ for the remaining admissible values of ω and j, depends only on σ and γ, and the correspondence $\sigma \to \tau$

is an isomorphic mapping of the group $\pi_n(K;\psi_1)$ onto the group $\pi_n(K;\psi_0)$. Therefore, by means of the isomorphisms corresponding to the distinguished one-dimensional cube γ, we can identify with one another all the groups $\pi_n(K;\psi)$. In view of this, we shall henceforth denote the groups $\pi_n(K;\psi)$ by $\pi_n(K)$. Here the group $\pi_1(K)$ is defined as the group of left operators of each of the groups $\pi_n(K)$, $n > 1$.

It is clear that our theorem will be proved if we construct a resolvent sequence $K_1, K_2, \dots$ and cubic mappings $w_i : K \to K_i$ such that, for every $i \geqslant 1$, the mapping w_i on the i-dimensional skeleton K^i of the complex K is isomorphic and coincides with the mapping w_{i+1}. We shall define the complex K_1 and the mapping w_1 by setting

$$K_1 = Q(\pi_1(K), 1)$$

and

$$w_1(\sigma)(\tau) = f_\sigma(\tau),$$

where σ is an arbitrary cube of the complex K, τ is an arbitrary one-dimensional cube of the complex I^n (n is the dimension of the cube σ), and f_σ is the cubic mapping $I^n \to K$ corresponding to the cube σ (see (1), p. 6). It is clear that the mapping w_1 is indeed isomorphic on the one-dimensional skeleton K^1. Moreover, it maps the two-dimensional skeleton K^2 onto the two-dimensional skeleton K_1^2. Suppose now that, for some $p \geqslant 1$, the complex K_p and the mapping w_p satisfying the conditions indicated above have already been constructed. Suppose, in addition, that w_p maps the $(p+1)$-dimensional skeleton K^{p+1} of the complex K onto the $(p+1)$-dimensional skeleton K_p^{p+1} of the complex K_p. Then there exists a cubic mapping

$$\bar{w}_p : K_p^{p+1} \to K^{p+1}$$

such that $w_p\bar{w}_p = 1$. Let now σ_1^j be an arbitrary $(p+2)$-dimensional cube of the complex K_p. Consider in the complex K the equation $\{\sigma_\omega^j\}$, where

$$\sigma_\omega^j = \bar{w}_p(\sigma\omega^j), \quad \omega = 0, 1; \quad 1 \leqslant j \leqslant p+2,$$

and, if $\omega = 0$, then $j \neq 1$. Let σ_0^1 be a solution of this $(p+2,0,1)$-equation. Then the solution of the $(p+2,0,2)$-equation $\{\tau_\omega^j\}$, where

$$\tau_0^1 = \sigma_0^1, \quad \tau_1^1 = \bar{w}_p(\sigma 0^1)$$

and

$$\tau_\omega^j = \sigma_0^1 \eta^1 \omega^j$$

for the remaining admissible values of ω and j, is a $(p+1)$-dimensional cube of the complex K belonging to the cube $\bar{w}_p(\sigma 0^{10^1})$. We shall denote the corresponding element of the group $\pi_{p+1}(K)$ by $k_p(\sigma)$. It turns out that the constructed function k_p is a cocycle of the complex K_p over the group $\pi_{p+1}(K)$ relative to the cocycle $1^{\pi}_{\pi_1(K)}$. Let K_{p+1} be the corresponding $(p+1)$-extension of the complex K_p. Then the mapping

$$w_{p+1} : K \to K_{p+1},$$

defined by the formula

$$w_{p+1}(\sigma) = (w_p(\sigma), c),$$

where σ is an arbitrary cube of the complex K, and c is a cubic cochain-function of height $p+1$ and dimension n (where n is the dimension of the cube σ) over the group $\pi_{p+1}(K)$, which assigns to an arbitrary $(p+1)$-dimensional cube τ of the complex I^n a solution of the $(p+2,0,2)$-equation $\{\sigma_\omega^j\}$, for which

$$\sigma_0^1 = w_p \bar{w}_p f_\sigma(\tau), \quad \sigma_1^1 = f_\sigma(\tau)$$

and

$$\sigma_\omega^j = f_\sigma(\tau) \eta^1 \omega^j$$

for the remaining admissible values of ω and j, satisfies, as is not hard to see, the conditions indicated above. Moreover, it maps the $(p+2)$-dimensional skeleton of the complex K onto the $(p+2)$-dimensional skeleton of the complex K_{p+1}. Thus the general step of the inductive construction is fully justified.

2. Minimal complexes. Let K be an arbitrary E-complex and ψ_0 some zero-dimensional cube of it. We shall call a subcomplex M of the complex K a **minimal subcomplex at the point** ψ_0 if the cube ψ_0 is the unique zero-dimensional cube contained in M, and if, for every cube $\sigma \in K$, all of whose faces lie in M, there exists one and only one cube in M homotopic to the cube σ. It is easy to see that at least one minimal subcomplex can be constructed for every zero-dimensional cube ψ_0 (one should bear in mind that distinguished cubes comparable with one another coincide). Moreover, any minimal subcomplex of the complex K has the same homotopy type (this notion for cubic complexes is defined in the obvious way) as the complex K itself, and any two minimal subcomplexes are isomorphic to one another. Since a minimal subcomplex is, obviously, an EN-complex, it follows that, by

of the theorem just proved, it is isomorphic to the limiting complex of a certain cubic resolvent. This resolvent is uniquely determined (up to isomorphism) by the complex K and is called the **cubic resolvent** of the E-complex K. It completely determines all homotopy (in particular, all homological) properties of this complex.

3. **Application to topological spaces.** Let X be an arbitrary linearly connected space. A **singular n-dimensional cube** of the space X is a continuous X-valued function $\sigma(t_1, \dots, t_n)$ of n real variables $t_1, \dots, t_n$, subject to the inequalities $0 \leq t_i \leq n$; $i = 1, \dots, n$ (as usual, X-valued functions of zero variables are regarded as points of the space X). It is easy to see that the set $Q(X)$ of all singular cubes of the space X is an E-complex with respect to the operators ε^i, $\varepsilon = 0, 1$, and η^i, defined by the formulas

$$(\sigma\varepsilon^i)(t_1, \dots, t_{n-1}) = \sigma(t_1, \dots, t_{i-1}, \varepsilon, t_i, \dots, t_{n-1}),$$

$$(\sigma\eta^i)(t_1, \dots, t_{n+1}) = \sigma(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}).$$

This complex is called the **singular cubic complex** of the space X, its minimal subcomplexes—the **minimal complexes** of the space X, and the corresponding resolvent—the **cubic resolvent** of the space X. From what was said above it is clear that the cubic resolvent completely describes the (singular) homotopy type of the space X (and, in particular, all its homology groups).

4. **Realization.** Transferring to the case of cubic complexes the known constructions for semisimplicial complexes, it is not difficult to show that for any EN-complex K there exists a space (indeed, a polyhedron) whose minimal complex is isomorphic to the complex K. In other words, any abstractly given cubic resolvent can be realized as the cubic resolvent of some polyhedron.

5. **Ω-pairs of resolvents.** As we have already noted, cubic resolvents are most convenient in the study of fiber spaces. As a first illustration of this general fact, let us find the relations that hold between the resolvents of a certain simply connected space and of the corresponding loop space.

Let $\{G_i, k_i\}$ and $\{H_i, l_i\}$ be two cubic resolvents, with $H_1 = 1$, and suppose that G_1 acts trivially on G_i for all $i > 1$. Suppose further that for any $i > 1$ a certain isomorphism $\varphi_i : G_i \to H_i$ is given. To each n-dimensional (n arbitrary) cochain function $c \in E(G_i, i)$ we assign an $(n+1)$-dimensional cochain function $\psi_i c \in E(H_{i+1}, i+1)$, putting, for any nondegenerate $(i+1)$-dimensional cube σ of the complex I^{n+1},

$$(\psi_i c)(\sigma) = \varphi_i c(\sigma'),$$

where σ' is the intersection of the cube σ, considered as a face of the unit cube of $(n+1)$-dimensional arithmetic space, with the unit cube of the n-dimensional coordinate hyperplane obtained by setting the last coordinate equal to zero (the intersection is regarded as zero if it is not an i-dimensional face). It is obvious that the mapping ψ_1 defines a certain dimension-raising-by-one mapping $\omega_1 : Q(G_1, 1) \to Q(H_2, 2)$. It turns out that if $\varphi_2^0 k_1 = \omega_1^* l_2$ (i.e. $\varphi_2 k_1(\sigma) = l_2(\omega_1 \sigma)$ for any $\sigma \in Q(G_1, 1)$), then the direct product $\omega_1 \times \psi_2$ of the mappings ω_1 and ψ_2 defines a certain (also dimension-raising-by-one) mapping $\omega_2 : K_2 \to L_3$, where $\{K_i\}$ and $\{L_i\}$ are the resolvent sequences corresponding to the resolvents $\{G_i, k_i\}$ and $\{H_i, l_i\}$. In general, if a mapping $\omega_i : K_i \to L_{i+1}$ has already been constructed and $\varphi_{i+1}^0 k_i = \omega_i^* l_{i+1}$, then the mapping

The mapping $\omega_i \times \varphi_{i+1}$ defines the mapping ω_{i+1}. If the mappings ω_i can be constructed in the described way for all $i \geqslant 1$, then we shall say that the given resolvents $\{G_i, k_i\}$ and $\{H_i, l_i\}$ form an Ω-pair. The resolvent $\{G_i, k_i\}$ will be called the first, and the resolvent $\{H_i, l_i\}$ the second component of the Ω-pair.

If the resolvents $\{G_i, k_i\}$ and $\{H_i, l_i\}$ form an Ω-pair, then the mappings $\omega_i : K_i \to L_{i+1}$ naturally define a certain mapping $\omega : K \to L$ of the limiting complexes. This mapping sends different cubes of the complex K to different cubes of the complex L, raises dimension by one, and satisfies the relations

$$\sigma \varepsilon^i \omega = \sigma \omega \varepsilon^i, \qquad \varepsilon = 0, 1, \quad i = 1, \dots, p,$$

$$\sigma \eta^j \omega = \sigma \omega \eta^j, \qquad j = 1, \dots, p + 1,$$

where σ is an arbitrary cube of the complex K, and p is its dimension. Moreover, a cube $\tau \in L$ then and only then has the form $\sigma\omega$, $\sigma \in K$, when the cubes $\tau \varepsilon^{p+1}$, where $p + 1$ is the dimension of τ, become, i.e. have the form $\psi_1 \eta^1 \dots \eta^1$, where ψ_0 is the (unique) zero-dimensional cube of the complex L. It turns out that the converse is also true, i.e. if for EN-complexes K and L there exists a mapping $\omega : K \to L$ satisfying the listed conditions, then the resolvents of these complexes form an Ω-pair.

The question of which resolvents can serve as components of Ω-pairs is very interesting. It is easy to see that any resolvent $\{H_i, l_i\}$ for which $H_1 = 1$ can be realized as the second component of a certain uniquely determined Ω-pair. Let $\{G_i, k_i\}$ be the first component of this Ω-pair. It turns out that if $\{H_i, l_i\}$ is the resolvent of some (connected) space X, then $\{G_i, k_i\}$ will be the resolvent of the loop space $\Omega(X)$, and the corresponding mapping ω of minimal complexes is determined by the well-known Hurewicz mapping

$$(\omega\sigma)(t_1, \dots, t_{p+1}) = \sigma(t_1, \dots, t_p)(t_{p+1}). \qquad \sigma \in \Omega(X),$$

Thus, the question of characterizing resolvents that can serve as primary components of Ω-pairs is equivalent to the question of characterizing (up to homotopy type) spaces that can serve as loop spaces.

Remark. Natural systems (i.e. "simplicial" resolvents) of loop spaces were studied by Aoki, Honma, and Kaneko (^{3}). In their work, in addition to results essentially coinciding with those set forth above, there is also a statement (Theorem 5) from which it follows directly that every homotopy-simple space has the (singular) homotopy type of some loop space. However, the simplest examples show that the latter assertion is manifestly false.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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