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# ON THE SECOND MIXED DERIVATIVE

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **ON THE SECOND MIXED DERIVATIVE**

*(Presented by Academician S. L. Sobolev on 1 VII 1958)*

Let  $E$  be some Banach space of functions on the two-dimensional torus\*, with  $D_\infty^{**}$  dense in  $E$ , and let it be known for a continuous function  $f(t, s)$  that  $\partial^2 f / \partial t^2$  and  $\partial^2 f / \partial s^2$  belong to  $E$ . Can one assert that  $\partial^2 f / \partial t \partial s$  also belongs to  $E$ ?

S. N. Bernstein showed<sup>1</sup> that when  $E$  is  $L^2$  or\*\*\*  $W$ , the answer is affirmative. For the case  $E = C$ , the question was not considered. In the present note it is shown that in this case the answer is negative.

For the proof, an example is given of a function  $F(t, s)$  such that  $\partial^2 F / \partial t^2$  and  $\partial^2 F / \partial s^2 \in C$ , but  $\partial^2 F / \partial t \partial s \notin C$ . In constructing the example the following theorem is essentially used:

**Petrini' s Theorem**<sup>2</sup>. Let  $\mu(\xi, \eta)$  be a continuous function;

$$H(t, s) = \int_{C^{2a}} \mu \log \frac{1}{r} d\xi d\eta$$

( $C^a$  is the disk of radius  $a$ ,  $4a < 1$ , with center at the point  $(0, 0)$ ;  $r = \sqrt{(\xi - t)^2 + (\eta - s)^2}$ \*\*\*\*).

Then inside  $C^{2a}$ ,  $H(t, s)$  is continuously differentiable. For the existence at a point  $(t, s)$  of the derivative  $\partial^2 H / \partial l_1 \partial l_2$  ( $l_1, l_2$  are directions making with the  $t$ -direction respectively the angles  $\psi_1, \psi_2$ ) it is necessary and sufficient that the limit exist

$$K_{l_1 l_2}(t, s) = \lim_{h \rightarrow 0} K_{l_1 l_2}^h(t, s); \quad K_{l_1 l_2}^h(t, s) = \int_{C^{2a} \setminus C_{ts}^h} \cos(2\nu - \psi_1 - \psi_2) \mu \frac{d\nu dr}{r} \quad (1)$$

( $C_{ts}^h$  is the disk of radius  $h$  with center at the point  $(t, s)$ ;  $r, \nu$  are polar coordinates with pole  $(t, s)$ ); moreover

$$\frac{\partial^2 H(t, s)}{\partial l_1 \partial l_2} = K_{l_1 l_2}(t, s) - \pi \mu(t, s) \cos(\widehat{l_1, l_2}). \quad (2)$$

**Lemma.** Let  $\mu = \frac{\cos 2\varphi}{\log(1/\rho)}$  ( $\rho, \varphi$  are polar coordinates with pole  $(0, 0)$ ).

Then

$$H(t, s) = \int_{C^{2a}} \mu \log \frac{1}{r} d\xi d\eta$$

has, for  $\rho \leq a$ , continuous  $\Delta H$  and  $\partial^2 H / \partial t \partial s$ , but  $\partial^2 H / \partial t^2$  and  $\partial^2 H / \partial s^2$  do not exist at the point  $(0, 0)$ .

\* For simplicity, doubly periodic functions are considered.

\*\*  $D_\infty$  is the space of all infinitely differentiable functions.

\*\*\*  $W$  is the space of functions with an absolutely convergent Fourier series

$$f(t, s) = \sum_{m, n=-\infty}^{\infty} a_{mn} e^{i(mt+ns)}$$

and norm

$$\|f\|_W = \sum_{m, n=-\infty}^{\infty} |a_{mn}|.$$

\*\*\*\* On the torus a metric is introduced in the natural way; distances less than one are thereby determined uniquely.

**Proof.**  $\Delta H = -2\pi\mu$ , and therefore is discontinuous.  $\partial^2 H / \partial t^2$  (as also  $\partial^2 H / \partial s^2$ ) does not exist at the point  $(0, 0)$ , since the condition of Petrini's theorem is not satisfied:

$$\int_0^{2\pi} \cos 2\nu d\nu \int_h^{2a} \frac{\cos 2\nu}{\log(1/r)} \frac{dr}{r} \rightarrow \infty \quad \text{as } h \rightarrow 0.$$

$\partial^2 H / \partial t \partial s$  for  $(t, s) \neq (0, 0)$  exists and is continuous, since  $\mu$  is continuously differentiable everywhere except at the point  $(0, 0)$ ; at the point  $(0, 0)$

$$K_{ts}^h = - \int_0^{2\pi} \sin 2\nu d\nu \int_h^{2a} \frac{\cos 2\nu}{\log(1/r)} \frac{dr}{r} = 0$$

for all  $h$ , so that  $\partial^2 H / \partial t \partial s$  exists at the point  $(0, 0)$  and is equal to zero. The continuity of  $\partial^2 H / \partial t \partial s$  at the point  $(0, 0)$  is verified directly. For this purpose, by Petrini's theorem, it is enough to show that  $K_{ts}(\varepsilon, \varphi_0) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , uniformly with respect to  $\varphi_0$ . We examine

$$-K_{ts}(\varepsilon, \varphi_0) = \lim_{h \rightarrow 0} \int_0^{2\pi} \int_0^{2a} \sin 2\nu \frac{\cos 2\varphi}{\log(1/\rho)} \frac{\rho d\rho d\varphi}{r^2},$$

where  $\rho \cos \varphi = r \cos \nu + \varepsilon \cos \varphi_0$ ,  $\rho \sin \varphi = r \sin \nu + \varepsilon \sin \varphi_0$ , and the prime sign means that on rays intersecting the circle  $C_{\varepsilon\varphi_0}^h$ , integration over the interval common with this circle is not performed.

We integrate by parts with respect to  $\rho$ ; for this we put

$$\frac{\sin 2\nu}{r^2} d\rho = d_\rho v, \quad \frac{\cos 2\varphi}{\log(1/\rho)} \rho = u.$$

Then

$$\begin{aligned} v &= \frac{\varepsilon \sin(\varphi - \varphi_0) - \rho \sin 2\varphi}{\rho^2 - 2\varepsilon\rho \cos(\varphi - \varphi_0) + \varepsilon^2}; \\ -K_{t,s}^h(\varepsilon, \varphi_0) &= \int_0^{2\pi} \left\{ uv \Big|_0^{\rho_h(\varphi)} + uv \Big|_{\rho_h(\varphi)}^{2a} - \int_0^{2a} v u'_\rho d\rho \right\} d\varphi \\ &= \int_0^{2\pi} uv \Big|_0^{2a} d\varphi - \int_\Delta uv \Big|_{\rho_h(\varphi)}^{\rho_h(\varphi)} d\varphi - \int_0^{2\pi} \int_0^{2a} v u'_\rho d\rho d\varphi, \end{aligned} \quad (3)$$

since  $\rho_h(\varphi) = \rho^h(\varphi)$  for  $\varphi \notin \Delta = (\varphi_0 - \arcsin \frac{h}{\varepsilon}, \varphi_0 + \arcsin \frac{h}{\varepsilon})$ ; in the interval  $\Delta$ ,  $\rho_h(\varphi)$  and  $\rho^h(\varphi)$  are determined as the roots of the equation  $r^2 = h^2$ .

$u(0,0) = 0$ , and therefore the first term in (3) is

$$g_1(\varepsilon, \varphi_0) = \int_0^{2\pi} \frac{\varepsilon \sin(\varphi + \varphi_0) - 2a \sin 2\varphi}{4a^2 - 4a\varepsilon \cos(\varphi - \varphi_0) + \varepsilon^2} \frac{2a \cos 2\varphi}{\log(1/2a)} d\varphi;$$

it is easy to see that, for small  $\varepsilon$ ,

$$|g_1(\varepsilon, \varphi_0)| < \frac{3\varepsilon}{a \log(1/2a)}. \quad (4)$$

The second term in formula (3) is

$$\begin{aligned} I_h(\varepsilon, \varphi_0) &= - \int_\Delta \left( \frac{\varepsilon \sin(\varphi + \varphi_0) - \rho \sin 2\varphi}{r^2} \frac{\rho}{\log(1/\rho)} \cos 2\varphi \right) \Big|_{\rho_h(\varphi)}^{\rho^h(\varphi)} d\varphi \\ &= - \frac{1}{h^2} \int_\Delta \left\{ \varepsilon \sin(\varphi + \varphi_0) \left[ \frac{\rho^h}{\log(1/\rho^h)} - \frac{\rho_h}{\log(1/\rho_h)} \right] \right. \\ &\quad \left. - \sin 2\varphi \left[ \frac{(\rho^h)^2}{\log(1/\rho^h)} - \frac{\rho_h^2}{\log(1/\rho_h)} \right] \right\} \cos 2\varphi d\varphi. \end{aligned} \quad (5)$$

we estimate as follows: since  $\rho^h - \rho_h \leq 2h$  and

$$\left(\frac{\rho}{\log(1/\rho)}\right)' = \frac{1}{\log(1/\rho)} + \frac{1}{\log^2(1/\rho)},$$

then, by Lagrange's theorem,

$$\frac{\rho^h}{\log(1/\rho^h)} - \frac{\rho_h}{\log(1/\rho_h)} \leq 2h \left( \frac{1}{\log(1/\xi)} + \frac{1}{\log^2(1/\xi)} \right) \quad (\rho_h < \xi < \rho^h),$$

and, consequently, does not exceed  $\frac{4h}{\log(1/\varepsilon)}$  [for sufficiently small  $\varepsilon$ ]. Analogously, for the second square bracket in (5) we obtain the estimate

$$\frac{(\rho^h)^2}{\log(1/\rho^h)} - \frac{\rho_h^2}{\log(1/\rho_h)} \leq \frac{6\varepsilon h}{\log(1/\varepsilon)}.$$

Then

$$|I_h(\varepsilon, \varphi_0)| \leq 2 \arcsin \frac{h}{\varepsilon} \cdot \frac{1}{h^2} \frac{10h}{\log(1/\varepsilon)} \leq \frac{10\pi}{\log(1/\varepsilon)}$$

and for  $I(\varepsilon, \varphi_0) = \lim I_h(\varepsilon, \varphi_0)$  we have

$$|I(\varepsilon, \varphi_0)| \leq \frac{10\pi}{\log(1/\varepsilon)}. \quad (6)$$

The third term in (3) is

$$J_h(\varepsilon, \varphi_0) = - \int_0^{2\pi} \int_0^{2a} \frac{\varepsilon \sin(\varphi + \varphi_0) - \rho \sin 2\varphi}{\rho^2 - 2\rho\varepsilon \cos(\varphi - \varphi_0) + \varepsilon^2} \left( \frac{1}{\log(1/\rho)} + \frac{1}{\log^2(1/\rho)} \right) \cos 2\varphi \, d\varphi \, d\rho.$$

We note that the integrand  $vu'_\rho$  has a single singularity of the first order at the point  $(\varepsilon, \varphi_0)$  and is integrable over the whole circle  $C^{2a}$ . Therefore

$$J(\varepsilon, \varphi_0) = \lim J_h(\varepsilon, \varphi_0) = - \int_{\rho \leq 2a} vu'_\rho \, d\varphi \, d\rho.$$

We use the formula

$$\frac{1}{1 - 2k \cos \alpha + k^2} = \frac{1}{1 - k^2} + \frac{2}{1 - k^2} \sum_{n=1}^{\infty} k^n \cos n\alpha$$

for  $|k| < 1$ , uniformly in  $\alpha$ . Expanding  $v$  in the corresponding series for  $\rho < \varepsilon$  and  $\rho > \varepsilon$  and integrating them, we obtain

$$J(\varepsilon, \varphi_0) = 4\pi \sin 4\varphi_0 \cdot \frac{1}{\varepsilon^4} \int_0^\varepsilon \rho^3 \left( \frac{1}{\log(1/\rho)} + \frac{1}{\log^2(1/\rho)} \right) d\rho,$$

whence

$$|J(\varepsilon, \varphi_0)| \leq \frac{8\pi}{\log(1/\varepsilon)}. \quad (7)$$

Inequalities (4), (6), and (7) give

$$|K_{ts}(\varepsilon, \varphi_0)| \leq \frac{20\pi}{\log(1/\varepsilon)}.$$

This completes the proof of the lemma.

Let  $e(\rho)$  be an infinitely differentiable function equal to one for  $\rho \leq a/2$  and to zero for  $\rho \geq a$ . Then  $\bar{H}(t, s) = H(t, s)e(\rho)$  is defined on the whole torus. It is not difficult to see that the function  $F(t, s) = \bar{H}(t + s, t - s)$  has continuous  $\partial^2 F / \partial t^2$  and  $\partial^2 F / \partial s^2$  on the whole torus, but  $\partial^2 F / \partial t \partial s$  does not exist at the point  $(0, 0)$ .

Thus it has been proved:

**Theorem.** The space  $B$ , obtained as the closure of  $D_\infty$  in the norm

$$\|f\| = \max_{t,s} \left\{ |f(t, s)|, \left| \frac{\partial f(t, s)}{\partial t} \right|, \left| \frac{\partial f(t, s)}{\partial s} \right|, \left| \frac{\partial^2 f(t, s)}{\partial t^2} \right|, \left| \frac{\partial^2 f(t, s)}{\partial s^2} \right| \right\}$$

is distinct from  $D_2$ —the space of twice continuously differentiable functions.

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*Note: Figure translations are in progress. See original paper for figures.*

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