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Abstract

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THEORY OF ELASTICITY

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STABILITY OF AN ORTHOTROPIC CYLINDRICAL SHELL UNDER TORSION WITH INTERNAL PRESSURE

(Presented by Academician Yu. N. Rabotnov on 12 VI 1958)

The problem of the stability of a thin orthotropic cylindrical shell under torsion with a sufficiently large internal pressure is solved. The shell is considered to be of "medium" length, i.e., to satisfy the condition

$$\varepsilon^{1/2} \max(K, K^{-1}) \ll (\pi R/l)^2 \ll \varepsilon^{-1/2} \min(K, K^{-1}), \quad (1)$$

where $\varepsilon = h^2/12R^2(1 - \nu_1\nu_2)$, $K = \sqrt{E_2/E_1}$; h, R, l are respectively the thickness, radius, and length of the shell; E_1, E_2 and ν_1, ν_2 are respectively the elastic moduli and Poisson ratios in the axial and circumferential directions ($E_1\nu_2 = E_2\nu_1$). It is assumed that the edges of the shell are hinged or clamped.

The results presented below are a generalization of the corresponding results obtained in ^(1,2) for an isotropic shell. The starting equations are the equilibrium equations of the shell with allowance for its deformation and the usual relations between internal force factors and deformations expressed through displacements. In the case when the forces T_1^0, T_2^0, S^0 (axial, circumferential, and shear forces corresponding to the fundamental form of equilibrium) do not depend on ξ and φ —dimensionless coordinates (see ⁽¹⁾)—one may seek the additional displacements in the form indicated in ⁽¹⁾, and, after eliminating from the initial equations the internal forces and moments corresponding to the additional displacements, carry out simplifications analogous to Donnell's simplifications for an isotropic shell of "medium" length. Then, in particular, for the additional displacement w in the radial direction one obtains the equation

$$\varepsilon \Delta_2 \Delta_1 w + \delta_{22} \frac{\partial^4 w}{\partial \xi^4} - \Delta_2 \left(t_1 \frac{\partial^2 w}{\partial \xi^2} + 2s \frac{\partial^2 w}{\partial \xi \partial \varphi} + t_2 \frac{\partial^2 w}{\partial \varphi^2} \right) = 0, \quad (2)$$

where $\Delta_i = \delta_{i1}\partial^4/\partial\xi^4 + \delta_i\partial^4/\partial\xi^2\partial\varphi^2 + \delta_{i2}\partial^4/\partial\varphi^4$ ($i = 1, 2$); $\delta_{11} = E_1/E_2$, $\delta_{12} = 1$, $\delta_1 = 4(1 - \nu_1\nu_2)G/E_2 + \nu_1E_1/E_2 + \nu_2$, $\delta_{21}E_2 = \delta_{22}E_1 = G$, $\delta_2 = 1 - G(\nu_1/E_2 + \nu_2/E_1)$ (G is the shear modulus); $t_1 = T_1^0/E_2h$, $s = S^0/E_2h$, $t_2 = T_2^0/E_2h$.

Let s_{cr} and $t_{2\text{cr}}$ be the critical values of s and t_2 when, respectively, only twisting moments M , applied to the edges of the shell, and only a uniform external pressure $q > 0$ act on the shell. From (2), in the same way as for an isotropic shell (see (2)), the formulas* are obtained

$$s_{\text{cr}} = M_{\text{cr}}/2\pi E_2 R^2 h = \pm 0.74(1 - \nu_1\nu_2)^{-5/8}(E_1/E_2)^{3/8} h R^{-1} \sqrt[4]{h R l^{-2}}, \quad (3)$$

$$-t_{2\text{cr}} = q_{\text{cr}} R/E_2 h = (2\pi/3\sqrt{6})(1 - \nu_1\nu_2)^{-3/4}(E_1/E_2)^{1/4} h l^{-1} \sqrt{h R^{-1}}. \quad (4)$$

* Cf. formulas (1.9) and (1.10) of paper (1). A misprint has crept into (1.10): instead of $t_{2\text{cr}}$ it should be $-t_{2\text{cr}}$.

Under the simultaneous action of prescribed twisting moments M_* and pressure q_* (external or internal), the question of the stability of the shell can be posed as was done in (1). Then this question reduces to determining the critical value λ_{cr} (the smallest eigenvalue) of the positive parameter λ under simple loading of the shell by moments $M = \lambda M_*$ and pressure $q = \lambda q_*$. Knowing how to determine λ_{cr} , it is easy to determine the critical load also in the case when the pressure remains constant during the loading process and only the moments M vary. Let us first turn to the case of simple loading. In this case

$$s = \lambda s_* = \lambda M_*/2\pi E_2 R^2 h, \quad t_2 = \lambda t_{2*} = -\lambda q_* R/E_2 h.$$

Of the boundary conditions, we shall satisfy only the following (for the satisfaction of the remaining conditions see below):

$$w = 0 \quad \text{for } \xi = \pm l/2R. \quad (5)$$

Then one may put

$$w = c_1 \sin(\mu_1 \xi - n\varphi) + c_2 \sin(\mu_2 \xi - n\varphi) \quad (6)$$

(n is a positive integer; c_1, c_2, μ_1, μ_2 are constants); in this case condition (5) becomes equivalent to the condition

$$\mu_2 - \mu_1 = m\delta \quad (\delta = 2\pi R/l, m = 1, 2, \dots). \quad (7)$$

Substituting into (2), in place of w , the right-hand side of equality (6) and replacing s and t_2 by λs_* and λt_{2*} ($t_1 = 0$), we obtain for the eigenvalue λ the formula:

$$\lambda = Y(\mu_i, n) = \frac{\varepsilon(\delta_{11}\mu_i^4 + \delta_1\mu_i^2n^2 + n^4)(\delta_{21}\mu_i^4 + \delta_2\mu_i^2n^2 + \delta_{22}n^4) + \delta_{21}\mu_i^4}{n(2s_*\mu_i - t_{2*}n)(\delta_{21}\mu_i^4 + \delta_2\mu_i^2n^2 + \delta_{22}n^4)} \quad (i = 1, 2),$$

where μ_1, μ_2 must satisfy condition (7). Hence it is clear that the eigenvalue λ is equal to the ordinate of a horizontal chord of length $m\delta$ of the curve $Y = Y(\mu, n) > 0$, where μ is variable and n is fixed (since $\lambda > 0$, only values $Y > 0$ are considered). The ordinate of the lowest of such chords, corresponding to all possible values of m and n , is equal to λ_{cr} . Let us introduce, instead of s_*, t_{2*}, μ , and n , new quantities by putting $s_* = s_*^0 s_{cr}$, $t_{2*} = t_{2*}^0 |t_{2*}|$,

$$\mu = (\pi R/l) M, \quad n = \sqrt[8]{12E_1(1 - \nu_1\nu_2)/E_2} \sqrt{\pi R/l} \sqrt[4]{R/h} N,$$

where s_{cr} and t_{2cr} are determined by formulas (3), (4), and the sign of s_{cr} is taken to be the same as that of s_* , so that $s_*^0 > 0$. Then the equality $Y = Y(\mu, n)$ is replaced by the following:

$$Y = Y(M, N) = \frac{(K^{-1}\gamma^4 M^4 + Q\gamma^2 M^2 N^2 + N^4)(\gamma^4 M^4 + L\gamma^2 M^2 N^2 + KN^4) + KM^4}{N(\alpha M - \beta N)(\gamma^2 M^4 + L\gamma^2 M^2 N^2 + KN^4)}, \quad (8)$$

where

$$\alpha = 3.94 s_*^0, \quad \beta = 1.76 t_{2*}^0, \quad \gamma^2 = \pi[12(1 - \nu_1\nu_2)]^{-1/4} Rl^{-1} (h/R)^{1/2},$$

$$Q = K^{1/2}[4G(1 - \nu_1\nu_2)/E_2 + \nu_1 K^{-2} + \nu_2], \quad L = K^{-1/2}(E_2/G - \nu_2 K^2 - \nu_1).$$

The eigenvalue λ can now be determined as the ordinate of a horizontal chord of length $2m$ of the curve $Y = Y(M, N) > 0$ for fixed N . The ordinate of the lowest of such chords, corresponding to all admissible values of m and N , is λ_{cr} . This circumstance makes it possible, for sufficiently large internal pressure q ($q < 0$; $t_{2*}, t_{2*}^0, \beta > 0$), to determine λ_{cr} with sufficient accuracy as the smallest value y_0 of a certain function. We shall show this by introducing, instead of M and N , new quantities

θ, η by means of the equalities

$$N = \theta M, \quad \eta = M^2(K^{-1}\gamma^4 + Q\gamma^2\theta^2 + \theta^4).$$

Then instead of (8) we shall have

$$Y = [\eta + r(\theta)\eta^{-1}]/\theta(\alpha - \beta\theta), \quad r(\theta) = \frac{\gamma^4 + QK\gamma^2\theta^2 + K\theta^4}{\gamma^4 + L\gamma^2\theta^2 + K\theta^4}. \quad (9)$$

From the known inequalities $\nu_1^2 < E_1/E_2$, $\nu_2^2 < E_2/E_1$ it follows that $\nu_1\nu_2 < 1$. Bearing in mind that $E_1, E_2, G > 0$, $\nu_1\nu_2 < 1$ and assuming $\nu_1, \nu_2 > 0$, we obtain $K, Q > 0$. Taking $L > 0$, i.e. $E_2/G > \nu_1 + \nu_2 E_2/E_1$, or $\nu_1 E_2/G > \nu_1^2 + \nu_2^2$, which is an insignificant restriction, we shall have $r(\theta) > 0$ for all real values of θ . Since $\alpha, \beta, Y > 0$, the domain D of variation of the quantities θ, η in (9) will be: $0 < \theta < \alpha/\beta$, $0 < \eta < +\infty$. Hence, and from the equality for $r(\theta)$, it is clear that for sufficiently large internal pressure (when $\alpha/\beta \ll 1$) the quantity $r(\theta)$ for all θ in the domain D will be close to unity and the quantity Y will differ little from the quantity

$$y = (\eta + \eta^{-1})/\theta(\alpha - \beta\theta) = J(\theta, \eta). \quad (10)$$

Let us note in passing that for any pressure the exact equality $Y = y$ is valid if $L = KQ$, i.e. if

$$G = \frac{1}{4}(1 - \nu_1\nu_2)^{-1} \left[\sqrt{(\nu_1 E_1 - \nu_2 E_2)^2 + 4E_1 E_2} - \nu_1 E_1 - \nu_2 E_2 \right]$$

(this formula is a generalization of the relation $G = E/2(1 + \nu)$ for an isotropic shell, into which it passes when $E_1 = E_2$, $\nu_1 = \nu_2$).

As for the isotropic shell (see (1)), the unique minimum $y_0 = 8\beta/\alpha^2$ of the function $y = J(\theta, \eta)$ in the domain D is the least value of this function in the domain D . To the value $y = y_0$ correspond the values $\theta = \theta_0 = \alpha/2\beta$, $\eta = \eta_0 = 1$ and the values

$$M = M_0 = (K^{-1}\gamma^4 + Q\gamma^2\theta_0^2 + \theta_0^4)^{-1/2}, \quad N = N_0 = \theta_0 M_0.$$

On passing to the variables M, N , formula (10) assumes the form

$$y = y(M, N) = \frac{(K^{-1}\gamma^4 M^4 + Q\gamma^2 M^2 N^2 + N^4) + M^4}{N(\alpha M - \beta N)(K^{-1}\gamma^4 M^4 + Q\gamma^2 M^2 N^2 + N^4)}.$$

Let λ_0 be the ordinate of the horizontal chord of length 2 which cuts off an arc of the curve $y = y(M, N_0)$ containing its lowest point (M_0, N_0) . The abscissa of the right end of the indicated chord will be $M_0 + \rho$, $0 < \rho < 2^*$. Put $c = \rho/M_0$. Then $\lambda_0 = y[M_0(1 + c), N_0]$ and**

$$0 < (\lambda_0 - y_0)/y_0 = \frac{1}{2}(\eta_* + \eta_*^{-1})[1 + c^2/(1 + 2c)] - 1,$$

$$\eta_* = M_0^2(1 + c^2) [K^{-1}\gamma^4 + Q\gamma^2\theta_0^2(1 + c)^{-2} + \theta_0^4(1 + c)^{-4}] = (1 + b)/(1 + c)^2,$$

$$b = \rho\gamma^2(2 + c)M_0 [K^{-1}\gamma^2(2 + 2c + c^2) + Q\theta_0^2] <$$

$$< (2 + c)(2 + 2c + c^2)\rho M_0(K^{-1}\gamma^4 + Q\gamma^2\theta_0^2 + \theta_0^4) = 4c + 6c^2 + 4c^3 + c^4.$$

Hence

$$\begin{aligned} 0 < (\lambda_0 - y_0)/y_0 &= \frac{b^2 - 4bc + 6c^2 + 4c^3 + c^4}{2(1 + 2c)(1 + b)} < \\ &< \frac{bc^2(6 + 4c + c^2) + 6c^2 + 4c^3 + c^4}{2(1 + 2c)(1 + b)} < 3c^2 + \frac{1}{2}c^4. \end{aligned} \quad (11)$$

Put $r(\theta) = 1 + \zeta(\theta)$. Obviously,

$$|\zeta| = |K\theta - L|\gamma^2\theta^2/(\gamma^4 + L\gamma^2\theta^2 + K\theta^4) <$$

$$< \zeta_1 = |KQ - L|\alpha^2/\beta^2\gamma^2 = 8|2G(1 - \nu_1\nu_2)/E_1 - E_2/2G + \nu_1 + \nu_2K^2| s_*^2/t_*^2,$$

where, if $\zeta \geq 0$ ($KQ - L \geq 0$), then $Y \leq y(1 + \zeta)$, and if $\zeta < 0$ ($KQ - L < 0$),

* It can be shown that, for sufficiently large internal pressure and a sufficiently small value of γ^2 , the quantity ρ will differ from unity by an arbitrarily small amount.

** The estimates given below were made by V. M. Darevskii. The estimate of the quantity $(\lambda_0 - y_0)/y_0$ is here somewhat improved in comparison with the corresponding estimate in work (1).

then $Y > y(1 + \zeta)$. Let λ_* be the ordinate of the horizontal chord of length 2 for the curve $Y = r(M, N_0)$ (λ_* is an eigenvalue of the parameter λ). It is clear that for $\zeta \geq 0$ we have $y_0 < \lambda_* \leq \lambda_0(1 + \zeta)$, and therefore (see (11)),

$$0 < (\lambda_* - y_0)/y_0 \leq (1 + \zeta)(\lambda_0 - y_0)/y_0 + \zeta < \zeta_* = \zeta_2(3 + \frac{1}{2}\zeta_2)(1 + \zeta_1) + \zeta_1,$$

where

$$\zeta_2 = c^2 = \rho^2/M_0^2 = K^{-1}\gamma^4\rho^2(1+\zeta_0), \quad \zeta_0 = (s_*/t_{2*})^2[4GE_2^{-1}(1-\nu_1\nu_2)+\nu_1+\nu_2K^2+K(s_*/t_{2*})^2].$$

For $\zeta < 0$ we have $y_0(1 + \zeta) < \lambda_* < \lambda_0$, and consequently, $(y_0 - \lambda_0)/y_0 < (y_0 - \lambda_*)/y_0 < |\zeta| < \zeta_1$, whence, taking (11) into account, $|y_0 - \lambda_*|/y_0 < \zeta_{**} = \max[\zeta_1, \zeta_2(3 + \frac{1}{2}\zeta_2)]$. It is obvious that for $\zeta \geq 0$ we have $\lambda_* \geq \lambda_{cr} > y_0$; consequently, $0 < \lambda_{cr} - y_0 \leq \lambda_* - y_0$, while for $\zeta < 0$ we have $\lambda_* \geq \lambda_{cr} > y_0(1 + \zeta)$; consequently, $-y_0|\zeta| < \lambda_{cr} - y_0 \leq \lambda_* - y_0$. Hence, for $\zeta \geq 0$ we obtain $0 < (\lambda_{cr} - y_0)/y_0 < \zeta_*$, and for $\zeta < 0$, $|\lambda_{cr} - y_0|/y_0 \leq \max(|\zeta|, |\lambda_* - y_0|/y_0) < \max[\zeta_1, \zeta_2(3 + \frac{1}{2}\zeta_2)]$, i.e., for $\zeta < 0$ we have $|\lambda_{cr} - y_0|/y_0 < \zeta_{**}$.

Thus, with a relative error less than ζ_* (for $KQ - L \geq 0$) or ζ_{**} (for $KQ - L < 0$), one may take $\lambda_{cr} = y_0 = 8\beta/\alpha^2$, or, substituting the values of α, β ,

$$\lambda_{cr} = 7.3\pi\sqrt{E_1E_2/(1-\nu_1\nu_2)}h^2R^4|q_*|M_*^{-2}. \quad (12)$$

From (12) it is easy to obtain a formula for the critical value of M (M_{cr}) under a sufficiently large internal pressure q that is constant during the loading process. Namely, multiplying both sides of equality (12) by λ_{cr} and replacing $\lambda_{cr}M_*$ by M_{cr} , and $\lambda_{cr}q_*$ by q , we obtain

$$M_{cr}^2 = 7.3\pi\sqrt{E_1E_2/(1-\nu_1\nu_2)}h^2R^4|q|. \quad (13)$$

Let us replace in the equalities for ζ_*, ζ_{**} the quantity $s_*/t_{2*} = \lambda_{cr}M_*/2\pi R^3|\lambda_{cr}q_*|$ by the quantity $M_{cr}/2\pi R^3|q|$, where M_{cr} is determined by formula (13), i.e., put $s_*/t_{2*} = 0.763[E_1E_2/(1-\nu_1\nu_2)]^{1/4}h/R|q|^{-1/2}$. Then the relative error in computing M_{cr} by formula (13) will obviously be less than ζ_* , if $KQ - L \geq 0$, or ζ_{**} , if $KQ - L < 0$.

Since the quantities ζ_0, ζ_1 tend to zero together with $s_*/t_{2*} = M_*/2\pi R^3|q_*|$, and $K^{-1}\gamma^4 = (\pi R/l)^2\varepsilon^{1/2}\sqrt{E_1/E_2} \ll 1$ by virtue of condition (1), then for a sufficiently large internal pressure (relative to atmospheric pressure it may be very small) the quantities ζ_*, ζ_{**} will be sufficiently small, i.e., formulas (12), (13) will be sufficiently accurate. Of course, these formulas can be used only when they lead to critical stresses smaller than the yield point.

As was established, $\lambda_{cr} \approx y_0 \approx \lambda_0$. But the value $\lambda = \lambda_0$ is attained at values M (M_1 and M_2) differing from M_0 by less than 2, while $M_0 \gg 1$. This circumstance makes it possible to show, in the same way as for an isotropic shell (see (1)), that under a sufficiently large internal pressure the fulfillment of condition (5)

practically leads to the automatic fulfillment of all the remaining boundary conditions, while the conditions of hinged support of the edges and of clamping become practically equivalent. Therefore one may consider that the solution presented satisfies all the boundary conditions of the given problem.

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