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Abstract

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INVERSE LIE THEOREMS FOR OPERATORS OF GENERALIZED TRANSLATION

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1. The present note is devoted to a partial converse of Theorems 2 and 3 of the note ([1]), which are an analogue of Lie' s second direct theorem for operators of generalized translation. We shall consider the cases indicated in the note ([1]) as cases I and II. The manifold V_n is assumed in this note to be analytic, as are all functions.

Recall that in case I there are $n(n+1)/2$ linearly independent infinitesimal operators $\widetilde{L}_{\alpha\beta;s}$ and the same number of operators $L_{\alpha\beta;t}$. In case II there are n infinitesimal operators of second order $\widetilde{N}_{\alpha;s}$ and the same number of operators $N_{\alpha;t}$.

Theorem 1. Let the operators $L_{\alpha\beta;t}$ satisfy the condition

$$L_{ij;t}L_{kl;t} - L_{i'j';t}L_{k'l';t} = \mu^{\alpha\beta}_{i'j'k'l'}L_{\alpha\beta;t}, \tag{1}$$

where $(i'j'k'l')$ is an arbitrary permutation of the indices $(ijkl)$ and $\mu^{\alpha\beta}_{i'j'k'l'}$ are constant numbers, and let the operators $\widetilde{L}_{\alpha\beta;s}$ satisfy the condition

$$\widetilde{L}_{kl;s}\widetilde{L}_{ij;s} - \widetilde{L}_{k'l';s}\widetilde{L}_{i'j';s} = \mu^{\alpha\beta}_{i'j'k'l'}\widetilde{L}_{\alpha\beta;s}. \tag{1'}$$

Suppose that the coefficients of the operators $L_{\alpha\beta;t}$ and $\widetilde{L}_{\alpha\beta;s}$ on the manifold V_n are analytic. Let $f(t)$ and $g_i(t)$ $(i = 1, 2, \dots, n)$ be arbitrary analytic functions on V_n. Then the Cauchy problem for the overdetermined system

$$\widetilde{L}_{\alpha\beta;s}u = L_{\alpha\beta;t}u \tag{2}$$

with initial conditions

$$u\big|_{s=0} = f(t); \tag{3}$$

$$\left.\frac{\partial u}{\partial s_i}\right|_{s=0} = g_i(t) \tag{4}$$

is solvable and, in the class of analytic functions, the solution of this problem is unique.

In the proof of this theorem it is first established that the successive derivatives of the function u with respect to the variables s_i are determined consistently from system (2). This follows from the commutation conditions (1) and (1′). After this, convergence of the series for $u(s,t)$ can be justified in the same way as in the proof of the classical Cauchy—Kovalevskaya theorem.

Theorem 2. Let, in the preceding theorem, $g_i(t) = h_i f(t)$, where h_i are constant numbers, and let the following conditions be satisfied:

1) $L_{\alpha\beta;s}\widetilde{L}_{\gamma\delta;s} = \widetilde{L}_{\gamma\delta;s}L_{\alpha\beta;s}$;
2) for any function $f(t)$

$$\widetilde{L}_{\alpha\beta;t}f(t)\big|_{t=0} = L_{\alpha\beta;t}f(t)\big|_{t=0},$$

$$\left.\frac{\partial}{\partial t_\nu}\widetilde{L}_{\alpha\beta;t}f(t)\right|_{t=0} = \left.\frac{\partial}{\partial t_\nu}L_{\alpha\beta}f(t)\right|_{t=0} \quad (\alpha,\beta,\nu = 1,2,\dots,n).$$

Then the solution of the system (3)—(4) (for $g_i(t) = h_i f(t)$) satisfies all the conditions of the generalized shift.

The space M from condition 3° (see (1)) is defined as the set of functions $f(t)$ satisfying the boundary conditions

$$\left.\frac{\partial}{\partial t_\nu}L_{\alpha_1\beta_1}L_{\alpha_2\beta_2}\cdots L_{\alpha_N\beta_N}f(t)\right|_{t=0} = h_\nu L_{\alpha_1\beta_1}L_{\alpha_2\beta_2}\cdots L_{\alpha_N\beta_N}f(t)\big|_{t=0}$$

for any integer $N \geqslant 0$.

If $\widetilde{L}_{\alpha\beta} = L_{\alpha\beta}$ and, consequently (see (1)), the operators $L_{\alpha\beta}$ commute, then the corresponding generalized-shift operators form a commutative family.

Theorem 2 reconstructs generalized-shift operators in the form of infinite power series. It is very interesting to clarify in what case the operators T^s admit a representation of the form (5) of the note ([1]).

2. In case II one can establish theorems analogous to Theorems 1 and 2.

Theorem 3. Let the operators $N_{\alpha;t}$ satisfy the commutation condition

$$N_\alpha N_\beta - N_\beta N_\alpha = c^\lambda_{\alpha\beta}N_\lambda, \tag{5}$$

and let the operators $\widetilde{N}_{\alpha;s}$ satisfy the commutation condition

$$\widetilde{N}_\beta\widetilde{N}_\alpha - \widetilde{N}_\alpha\widetilde{N}_\beta = c^\lambda_{\alpha\beta}\widetilde{N}_\lambda. \tag{5'}$$

Let the coefficients of the operators $N_{\alpha;t}$ and $\widetilde{N}_{\alpha;s}$ on the manifold V_n be analytic. In this case the Cauchy problem

$$\widetilde{N}_{\alpha;s}u = N_{\alpha;t}u; \tag{6}$$

$$u\big|_{s=0} = f(t); \tag{7}$$

$$\left.\frac{\partial^\lambda u}{\partial s_1^{\lambda_1}\cdots\partial s_n^{\lambda_n}}\right|_{s=0} = g_{\lambda_1,\dots,\lambda_n}(t), \tag{8}$$

where $0 \leqslant \lambda_j \leqslant 1$ $(j = 1, 2, \dots, n)$, $\lambda = \lambda_1 + \dots + \lambda_n$, for analytic $f(t)$ and $g_{\lambda_1,\dots,\lambda_n}(t)$, is solvable in the class of analytic functions and the solution is unique.

Theorem 4. Let

$$g_{\lambda_1,\dots,\lambda_n}(t) = h_{\lambda_1,\dots,\lambda_n} f(t),$$

where $h_{\lambda_1,\dots,\lambda_n}$ are constant numbers, and suppose that the following conditions are satisfied:

1) $N_{\alpha;s}\widetilde{N}_{\beta;s} = \widetilde{N}_{\beta;s}N_{\alpha;s}$;
2) for any function $f(t)$

$$\widetilde{N}_{\alpha;t}f(t)\big|_{t=0} = N_{\alpha;t}f(t)\big|_{t=0}; \qquad \left.\frac{\partial^\lambda}{\partial t_1^{\lambda_1}\cdots\partial t_n^{\lambda_n}}\widetilde{N}_{\alpha;t}f(t)\right|_{t=0} = \left.\frac{\partial^\lambda}{\partial t_1^{\lambda_1}\cdots\partial t_n^{\lambda_n}}N_{\alpha;t}f(t)\right|_{t=0}.$$

Then the solution of the system (6), (7), (8) (for

$$g_{\lambda_1,\dots,\lambda_n}(t) = h_{\lambda_1,\dots,\lambda_n} f(t)$$

) satisfies all the conditions of the generalized shift.

The space M from condition 3° (see (1)) is defined as the set of functions satisfying the boundary conditions

$$\left.\frac{\partial^\lambda}{\partial s_1^{\lambda_1}\cdots\partial s_n^{\lambda_n}}N_{\alpha_1}N_{\alpha_2}\cdots N_{\alpha_N}f(t)\right|_{t=0} = h_{\lambda_1,\dots,\lambda_n}N_{\alpha_1}N_{\alpha_2}\cdots N_{\alpha_N}f(t)\big|_{t=0}$$

for any integer $N \geqslant 0$.

If $\tilde{N}_{\alpha;t} = N_{\alpha;t}u$, and, consequently, the operators $N_{\alpha;t}$ commute, then the corresponding generalized translation operators form a commutative family.

3. The compatibility conditions for systems (2) and (6) obtained by us in Theorems 1 and 3 make it possible to investigate finite-dimensional invariant subspaces for noncommuting systems of differential operators satisfying the commutation conditions (1) or (5). We shall formulate the results obtained by us in this direction in the form of two theorems.

Theorem 5. Let the operators $L_{ij;t}$ satisfy condition (1), and let the constant matrices

$$A_{ij} = \left(A^{\lambda}_{ij;\alpha}\right)^{N}_{\alpha,\lambda=1} \quad (i,\ j = 1,\ 2, \dots,\ n;\ A_{ij} = A_{ji})$$

satisfy the relations

$$A_{ij}A_{kl} - A_{i'j'}A_{k'l'} = \mu^{\alpha\beta}_{i'j'k'l'}A_{\alpha\beta}.$$

Then the system of equations

$$L_{ij;t}E(t) = A_{ij}E(t),$$

where $E(t)$ is a matrix-function of order N, with the initial conditions

$$E(0) = E, \qquad \frac{\partial E}{\partial t_\nu} = F_\nu,$$

where E is the identity matrix and F_ν are arbitrary constant matrices, is compatible and the solution is unique (in the class of analytic functions).

If conditions 1) and 2) of Theorem 2 are satisfied and the matrices F_ν commute with the matrices A_{ij} (this condition is satisfied if $F_\nu = h_\nu E$, where h_ν are constants and E is the identity matrix), then

$$\tilde{L}_{ij;s}E(s) = E(s)A_{ij}.$$

Theorem 6. Let the operators $N_{\alpha;t}$ $(\alpha = 1, 2, \dots, n)$ satisfy condition (5), and let the constant matrices

$$A_i = \left(A^{\lambda}_{i,\alpha}\right)^{N}_{\alpha,\lambda=1} \quad (i = 1,\ 2, \dots,\ n)$$

satisfy the relations

$$A_{iA}k - A_{kA}i = c^{\lambda}_{ik}A_{\lambda}.$$

Then the system of equations

$$N_{i;t}E(t) = A_{iE}(t),$$

where $E(t)$ is a matrix-function of order N, with the initial conditions

$$E(0) = E, \qquad \left.\frac{\partial^\lambda E}{\partial t_1^{\lambda_1} \dots \partial t_n^{\lambda_n}}\right|_{t=0} = F_{\lambda_1,\dots,\lambda_n},$$

where E is the identity matrix, $F_{\lambda_1,\dots,\lambda_n}$ are arbitrary constant matrices, $0 \leqslant \lambda_j \leqslant 1$ $(j = 1, 2, \dots, n)$, is compatible and the solution is unique (in the class of analytic functions).

If conditions 1) and 2) of Theorem 4 are satisfied and the matrices $F_{\lambda_1,\dots,\lambda_n}$ commute with the matrices A_i, then

$$\tilde{N}_{i;s} E(s) = E(s) A_i.$$

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References Cited

1. B. M. Levitan, *DAN*, **123**, No. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.