MATHEMATICS

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Abstract

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ON THE NUMBER OF SOLUTIONS OF A HOMOGENEOUS SINGULAR INTEGRAL EQUATION WITH CONTINUOUS COEFFICIENTS

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1. Let Γ be a contour consisting of a finite number of simple smooth closed oriented curves with continuous curvature. Denote by $L_2(\Gamma)$ the space of complex-valued functions with summable square, given on Γ. The norm in $L_2(\Gamma)$ is defined by the equality

$$\|\varphi\| = \left(\int_\Gamma |\varphi(t)|^2\,|dt|\right)^{1/2}.$$

S. G. Mikhlin showed ([1]) that the singular integral equations of the form

$$A\varphi = a(t)\varphi(t) - \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t) \qquad (t \in \Gamma), \tag{1}$$

considered in the space $L_2(\Gamma)$, under the conditions that: a) the coefficients $a(t)$ and $b(t)$ are continuous functions; b) $a^2(t) - b^2(t) \neq 0$ $(t \in \Gamma)$, are subject to F. Noether' s theorems. These theorems had previously been proved ([1,2]) only for the case when the functions $a(t)$ and $b(t)$ satisfy the Hölder condition and condition b).

F. Noether' s theorems for equation (1) consist in the fact that, when conditions a) and b) are satisfied, the equation $A\varphi = f$ is normally solvable, the homogeneous equations $A\varphi = 0$ and $A^*\psi = 0$ have finite numbers of linearly independent solutions, and the index $\varkappa(A)$ of the operator A is computed by the formula*

$$\varkappa(A) = \frac{1}{2\pi}\int_\Gamma dt\,\arg c(t), \tag{2}$$

where $c(t) = (a(t) + b(t))(a(t) - b(t))^{-1}$.

Let us note that the index of the operator A is the difference $\alpha(A) - \alpha(A^*)$ ($= \varkappa(A)$), where the numbers $\alpha(A)$ and $\alpha(A^*)$ denote, respectively, the numbers of linearly independent solutions of the equations $A\varphi = 0$ and $A^*\psi = 0$.

For the case when the functions $a(t)$ and $b(t)$ satisfy the Hölder condition and condition b), the following proposition is known (see ([2]) and ([3]), p. 47).

The homogeneous integral equation $A\varphi = 0$ has exactly $\varkappa(A)$ linearly independent solutions for $\varkappa(A) > 0$ and only the trivial solution for $\varkappa(A) \leq 0$.

The proof of this proposition is essentially based on the fact that, in the case of Hölder coefficients, the equation

$$a(t)\varphi(t) - \frac{b(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} \, d\tau = 0 \tag{3}$$

is solved effectively ([2,3]).

It turns out that in the case of continuous coefficients $a(t)$ and $b(t)$ (when, generally speaking, there is no method for effectively solving equation (3)), the proposition formulated above remains valid. A proof of this fact will be given below.

For simplicity, suppose that the contour Γ bounds on the left a simply connected bounded domain containing the origin.

Every solution $\varphi(t)$ of equation (3), by the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)}{t - z} \, dt \qquad (\varphi \in L_2(\Gamma);\ t \in \Gamma) \tag{4}$$

generates a piecewise holomorphic function whose boundary values $\Phi_+(t)$ and $\Phi_-(t)$ satisfy almost everywhere on Γ the relation (Hilbert problem)

$$c(t)\Phi_-(t) = \Phi_+(t). \tag{5}$$

Conversely, every piecewise holomorphic function $\Phi(z)$ of the form (4) that is a solution of the Hilbert problem (5) has the property that the function $\varphi(t)$ is a solution of equation (3).

From what has been said it follows, in particular, that the number of linearly independent solutions of the Hilbert problem (5) of the form (4) coincides with the number of linearly independent solutions of equation (3) in the class $L_2(\Gamma)$.

Let us note two simple properties of solutions of the Hilbert problem (5).

1°. If a function $\Phi(z)$ of the form (4), which is a solution of the Hilbert problem (5), has order $k(> 0)$ at infinity, then the functions

$$z^m\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{t^m\varphi(t)}{t-z}\,dt \qquad (z\in\Gamma;\quad m=0,1,\dots,k-1)$$

are also solutions of the Hilbert problem (5).

2°. The order of the zero at infinity of any solution of the Hilbert problem (5) of the form (4) does not exceed the number α of linearly independent solutions of equation (3).

Lemma. Let $n(\le\alpha)$ be the greatest order at infinity of solutions of the form (4) of the Hilbert problem (5), and let the solution

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)}{t-z}\,dt \qquad (z\in\Gamma,\ \varphi\in L_2(\Gamma))$$

of problem (5) have order n at infinity. Then the numbers n and α are equal, and the functions

$$\varphi(t),\ t\varphi(t),\dots,\ t^{n-1}\varphi(t) \qquad (\in L_2(\Gamma))$$

form a basis of the set of all solutions of equation (3).

Theorem 1. If the coefficients $a(t)$ and $b(t)$ are continuous functions and the difference $a^2(t)-b^2(t)$ vanishes nowhere on Γ, then the homogeneous equation (3) has exactly

$$\varkappa(A) = \frac{1}{2\pi}\int_\Gamma dt\ \arg\frac{a(t)+b(t)}{a(t)-b(t)}$$

linearly independent solutions when $\varkappa(A)>0$, and only the trivial solution when $\varkappa(A)\leqslant 0$.

Proof. Suppose that the numbers $\alpha(A)$ and $\alpha(A^*)$ are simultaneously nonzero. Then the equation

$$A_1\chi = a_1(t)\chi(t) - \frac{b_1(t)}{\pi i}\int_\Gamma \frac{\chi(\tau)}{\tau-t}\,d\tau = 0 \qquad (\chi\in L_2(\Gamma)), \tag{6}$$

where

$$a_1(t) = \frac{1}{2}(a(t)(t+1)-b(t)(t-1));$$

$$b_1(t) = \frac{1}{2}(a(t)(t-1)-b(t)(t+1)),$$

has $\alpha(A)+1$ linearly independent solutions. Indeed, a solution of equation (6) leads to the Hilbert problem

$$c_1(t)X_-(t) = X_+(t), \tag{7}$$

where $c_1(t) = tc(t)$. Consequently, if a function $\Phi(z)$ of the form (4), having order $\alpha(A)$ at infinity, is a solution of problem (5), then the function

$$X(z) = \frac{1}{2\pi i}\int_\Gamma \frac{X_+(t) - t^{-1}X_-(t)}{t-z}\,dt \qquad (t \in \Gamma)$$

is a solution of problem (7) and has order $\alpha(A)+1$ at infinity, i.e. $\alpha(A_1) \geqslant \alpha(A)+1$ ($\geqslant 2$).

Conversely, if a function $X(z)$ of the form (4), having order $\alpha(A_1)$ ($\geqslant 2$) at infinity, is a solution of problem (7), then the function

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{X_+(t) - X_-(t)}{t-z}\,dt \qquad (z \notin \Gamma)$$

has order $\alpha(A_1)-1$ at infinity and is a solution of the Hilbert problem (5), i.e. $\alpha(A) \geqslant \alpha(A_1)-1$. Thus, $\alpha(A_1) = \alpha(A)+1$.

The equation $A_1^*\psi = 0$ (see (3), p. 49) has the same number of linearly independent solutions as the equation

$$a_1(t)\psi(t) + \frac{b_1(t)}{\pi i}\int_\Gamma \frac{\psi(\tau)}{\tau - t}\,d\tau = 0. \tag{8}$$

Comparing the numbers of linearly independent solutions of equation (8) and the equation $A^*\psi = 0$ in the manner indicated above, and taking into account here that $\alpha(A^*) \geqslant 1$, we obtain

$$\alpha(A_1^*) = \alpha(A^*) - 1.$$

It follows that, on the one hand,

$$\varkappa(A_1) = \alpha(A_1) - \alpha(A_1^*) = \alpha(A) + 1 - \alpha(A^*) + 1 = \varkappa(A) + 2;$$

on the other hand, the relation $c_1(t) = tc(t)$ and formula (2) for computing the index entail the equality $\varkappa(A_1) = \varkappa(A)+1$. Thus, if the conditions of the theorem hold, then at least one of the numbers $\alpha(A)$ and $\alpha(A^*)$ is equal to zero.

If we now suppose that $\varkappa(A) > 0$, then $\alpha(A)$ is a positive number; consequently, $\alpha(A^*) = 0$ and $\alpha(A) = \varkappa(A)$. If, however, $\varkappa(A) \leqslant 0$, then $\alpha(A^*) > 0$ and, consequently, $\alpha(A) = 0$. The theorem is proved.

2. **Theorem 2.** Let the function $k(t) \in L_2(-\infty, \infty)$ satisfy the conditions: a) the Fourier transform $K(\lambda)$ of the function $k(t)$ is -

continuous function tending to zero as $t \to \infty$; b) $1 - K(\lambda) \neq 0$ $(-\infty < \lambda < \infty)$.

Then the equation

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds = 0 \qquad (0 \leq t < \infty) \tag{9}$$

has in the space $L_2(0,\infty)$ exactly

$$\nu = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d_\lambda \arg(1 - K(\lambda))$$

linearly independent solutions for $\nu > 0$, and the unique zero solution for $\nu \leq 0$.

Theorem 3. Suppose that the series

$$a(t) = \sum_{k=-\infty}^{\infty} a_k t^k \qquad (|t| = 1)$$

converges uniformly on the unit circle and that the function $a(t)$ does not vanish anywhere on the unit circle.

Then the system of equations

$$\sum_{j=0}^{\infty} a_{k-j}\xi_j = 0 \qquad (k = 0, 1, 2, ...) \tag{10}$$

has in the space l_2 exactly

$$\nu = -\frac{1}{2\pi}\int_{|t|=1} d_t \arg a(t)$$

linearly independent solutions for $\nu > 0$, and the unique trivial solution for $\nu \leq 0$.

The validity of Theorems 3 and 4 follows from Theorem 1 and from the relation established between the numbers of linearly independent solutions of equations (9), (10) and the numbers of linearly independent solutions of specially chosen singular integral equations of the form (3) (see ([4])).

A proposition differing from Theorem 2 only in that, besides condition b), the sole condition imposed on the kernel $k(t)$ is $k(t) \in L_1(-\infty, \infty)$, was proved by

M. G. Krein in ([5]) for an entire class of spaces $E(0,\infty)$, among which $L_2(0,\infty)$ is included (in ([5]) a detailed bibliography on equations (9), (10) is given).

An analogous remark can also be made with respect to Theorem 3, which is a discrete analogue of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.