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# EXTREMAL PROBLEMS IN THE CLASS OF TRIGONOMETRIC POLYNOMIALS

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **EXTREMAL PROBLEMS IN THE CLASS OF TRIGONOMETRIC POLYNOMIALS**

*(Presented by Academician S. N. Bernstein on 17 III 1958)*

Many problems in the theory of approximation of periodic functions by means of trigonometric polynomials are connected with extremal properties of these polynomials.

An arbitrary trigonometric polynomial of order  $(n_1, n_2, \dots, n_k)$  has the form

$$T_{n_1, \dots, n_k}(x_1, x_2, \dots, x_k) = \sum_{\nu_1 = -n_1}^{n_1} \cdots \sum_{\nu_k = -n_k}^{n_k} C_{\nu_1, \dots, \nu_k} e^{i\nu_1 x_1 + i\nu_2 x_2 + \cdots + i\nu_k x_k}.$$

We introduce the notation:

$$\|T_{n_1, \dots, n_k}\|_p = \left( \int_0^{2\pi} \cdots \int_0^{2\pi} |T_{n_1, \dots, n_k}(t_1, \dots, t_k)|^p dt_1 \cdots dt_k \right)^{1/p},$$

where  $p \geq 1$  is an arbitrary number.

Among the numerous investigations in the class of trigonometric polynomials we note the well-known inequality of S. N. Bernstein <sup>(1)</sup>, p. 26, and its generalization, obtained by N. K. Bari <sup>(3)</sup> in the space  $L_p(a, b)$ , where  $0 < a < b < 2\pi$ . A remarkable inequality was obtained by S. M. Nikol'skii <sup>(4)</sup> for trigonometric polynomials of many variables:

$$\|T_{n_1, \dots, n_k}\|_{p'} \leq 2^k \left( \prod_{j=1}^k n_j \right)^{1/p-1/p'} \|T_{n_1, \dots, n_k}\|_p \quad (1 \leq p \leq p'). \quad (1)$$

In this direction a number of general problems were first considered by S. N. Bernstein (see, for example, <sup>(1)</sup>, pp. 97–100; <sup>(2)</sup>, pp. 426–432 and 605–606).

We assume that the function  $K(x_1, x_2, \dots, x_k)$  is integrable inside the  $k$ -dimensional cube  $(G_k) = (0 \leq x_j \leq 2\pi, 1 \leq j \leq k)$ , and consider the functional

$$J_k(T_{n_1, \dots, n_k}) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \dots \int_0^{2\pi} T_{n_1, \dots, n_k}(t_1, \dots, t_k) K(t_1, \dots, t_k) dt_1 \dots dt_k.$$

With respect to this functional the following proposition holds.

**Theorem 1.** If  $T_{n_1, \dots, n_k}(x_1, \dots, x_k)$  is an arbitrary trigonometric polynomial of order  $(n_1, n_2, \dots, n_k)$ , and if  $K(x_1, \dots, x_k)$  is an integrable function inside the  $k$ -dimensional cube  $(G_k) = (0 \leq x_j \leq 2\pi, 1 \leq j \leq k)$ , with Fourier coefficients  $b_{\nu_1, \nu_2, \dots, \nu_k}$  ( $\nu_j = 0, \pm 1, \dots; 1 \leq j \leq k$ ), then for the functional  $J_k(T_{n_1, \dots, n_k})$  the following inequality holds:

$$|J_k(T_{n_1, \dots, n_k})| \leq (2\pi)^{-k/p} \|T_{n_1, \dots, n_k}\|_p \left( \sum_{-n_1}^{n_1} \dots \sum_{-n_k}^{n_k} |b_{\nu_1, \dots, \nu_k}|^p \right)^{1/p}.$$

for any  $p$  satisfying the condition  $1 \leq p \leq 2$ .

We note several consequences of Theorem 1 for a specifically given  $K(x_1, \dots, x_k)$ .

1. For an arbitrary trigonometric polynomial  $T_{n_1, \dots, n_k}(x_1, \dots, x_k)$ , the inequality holds

$$\max_{\substack{0 \leq x_j \leq 2\pi \\ (1 \leq j \leq k)}} |T_{n_1, \dots, n_k}(x_1, \dots, x_k)| \leq \prod_{j=1}^k \left( \frac{2n_j + 1}{2\pi} \right)^{1/p} \|T_{n_1, \dots, n_k}\|_p.$$

2. For the derivatives of the trigonometric polynomial  $T_{n_1, \dots, n_k}(x_1, \dots, x_k)$ , for  $1 \leq p \leq 2$  we have:

$$\max_{\substack{0 \leq x_j \leq 2\pi \\ (1 \leq j \leq k)}} \left| \frac{\partial^{m_1 + \dots + m_k} T_{n_1, \dots, n_k}}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right| \leq \pi^{-k/p} \prod_{j=1}^k \frac{n_j^{m_j+1/p}}{(pm_j + 1)^{1/p}} \cdot \|T_{n_1, \dots, n_k}\|_p (1+o(1)).$$

3. Let real or complex numbers  $\lambda_{s_1, \dots, s_k}$  be given ( $s_j = 0, \pm 1, \pm 2, \dots, \pm n_j; 1 \leq j \leq k$ ), and let a trigonometric polynomial  $T_{n_1, \dots, n_k}(x_1, \dots, x_k)$  have coefficients  $C_{\nu_1, \dots, \nu_k}$  ( $\nu_j = 0, \pm 1, \pm 2, \dots, \pm n_j; 1 \leq j \leq k$ ). Then for the trigonometric sum

$$f_\lambda(x_1, \dots, x_k) = \sum_{s_1=-n_1}^{n_1} \dots \sum_{s_k=-n_k}^{n_k} \lambda_{s_1, \dots, s_k} C_{s_1, \dots, s_k} e^{i(s_1 x_1 + \dots + s_k x_k)}$$

for  $1 \leq p \leq 2$  the inequality holds

$$\max_{\substack{0 \leq x_j \leq 2\pi \\ (1 \leq j \leq k)}} |f_\lambda(x_1, \dots, x_k)| \leq (2\pi)^{-k/p} \|T_{n_1, \dots, n_k}\|_p \left( \sum_{-n_k}^{n_k} \dots \sum_{-n_1}^{n_1} |\lambda_{s_1, \dots, s_k}|^p \right)^{1/p}.$$

4. If the polynomial  $\tilde{T}(x)$  is conjugate to the polynomial  $T(x)$ , then for  $1 \leq p \leq 2$  we have

$$\max_{0 \leq x \leq 2\pi} |\tilde{T}(x)| \leq 2 \left( \frac{n}{2\pi} \right)^{1/p} \|T\|_p,$$

$$\max_{0 \leq x \leq 2\pi} |\tilde{T}'(x)| \leq 2 \left( \frac{n}{2\pi(p+1)} \right)^{1/p} n \|T\|_p (1 + o(1)).$$

**Theorem 2.** If  $1 \leq p \leq 2$ ,  $p' \geq p$ , then for an arbitrary trigonometric polynomial  $T_{n_1, \dots, n_k}(x_1, \dots, x_k)$  the inequality holds

$$\|T_{n_1, \dots, n_k}\|_{p'} \leq \left( \prod_{j=1}^k \frac{2n_j + 1}{2\pi} \right)^{1/p-1/p'} \|T_{n_1, \dots, n_k}\|_p. \quad (2)$$

We observe that inequality (2) sharpens the inequality of S. M. Nikol'skii (1). Similar assertions hold for algebraic polynomials of the form

$$Q(x_1, \dots, x_k) = \sum_{\nu_1=0}^{n_1} \dots \sum_{\nu_k=0}^{n_k} a_{\nu_1, \dots, \nu_k} x_1^{\nu_1} \dots x_k^{\nu_k}.$$

**Theorem 3.** For the algebraic polynomial  $Q(x_1, \dots, x_k)$  in the  $k$ -dimensional cube  $(W_k) = (a \leq x_j \leq b, 1 \leq j \leq k)$ , for  $1 \leq p \leq 2$  the inequalities hold

$$\max_{\substack{a \leq x_j \leq b \\ (1 \leq j \leq k)}} \left| \frac{\partial Q(x_1, \dots, x_k)}{\partial x_j} \right| \leq \left( \frac{2}{\pi} \right)^{k/p} \frac{n_j^{1+1/p}}{\sqrt{(x_j - a)(b - x_j)}} \left( \prod_{\substack{l=1 \\ l \neq j}}^k n_l^{1/p} \right) L_{k,p}(Q),$$

$$\max_{\substack{a \leq x_j \leq b \\ (1 \leq j \leq k)}} |Q(x_1, x_2, \dots, x_k)| \leq \prod_{j=1}^k \left( \frac{2n_j + 1}{\pi} \right)^{1/p} L_{k,p}(Q),$$

where

$$L_{k,p}(Q) = \left( \int_a^b \dots \int_a^b |Q(x_1, \dots, x_k)|^p \frac{dx_1 dx_2 \dots dx_k}{\prod_{j=1}^k (x_j - a)(b - x_j)} \right)^{1/p}.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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