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Abstract

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ON THE TWO-DIMENSIONAL THIRRING MODEL

(Presented by Academician N. N. Bogolyubov, 5 V 1958)

Following Thirring ([1]), let us consider in two-dimensional space-time a nonlinear theory of a spinor field with the interaction Lagrangian

$$\mathcal{L}(x) = g : \bar{\psi}(x)\sigma^n\psi(x)\bar{\psi}(x)\sigma^n\psi(x) :, \tag{1}$$

where $\sigma^0 = I$, and $\sigma^1, \sigma^2, \sigma^3$ are the usual Pauli matrices of the second rank; the summation in (1) is defined as follows:

$$\sigma^n \times \sigma^n = I \times I - \sigma^1 \times \sigma^1 - \sigma^2 \times \sigma^2 - \sigma^3 \times \sigma^3. \tag{2}$$

The Lagrangian (1) is the unique combination symmetric with respect to interchange of two anticommuting operators ψ and (or) two $\bar{\psi}$. It can be reduced to the form

$$\mathcal{L}(x) = 4g : \bar{\psi}(x)\psi(x)\bar{\psi}(x)\psi(x) : .$$

Let us consider the element of the S-matrix corresponding to the scattering of two ψ-particles of zero mass, which can be written in the form

$$S = \frac{ig}{4\pi^2} \int \bar{\psi}_\chi(p')\psi_\beta(q)\bar{\psi}_\gamma(q')\psi_\delta(p)\, \delta(p' + q' - p - q) \times$$

$$\times \Gamma_{\alpha\beta,\gamma\delta}(p', q', p, q,)\, d^2p'\, d^2p\, d^2q'\, d^2q, \tag{3}$$

where the function Γ possesses the obvious antisymmetry properties, and summation over repeated indices is understood.

In the second order of perturbation theory we obtain for Γ the following expression

$$-\frac{2g}{\pi}(\sigma^n_{\alpha\beta} \times \sigma^n_{\gamma\delta}) \ln \frac{P^2}{Q^2} - \frac{g}{\pi} \frac{\{\hat{P}_{\alpha\beta} \times \hat{P}_{\gamma\delta} + \hat{P}_{\alpha\delta} \times \hat{P}_{\gamma\beta}\}}{P^2} \cdot C, \tag{4}$$

where $P = (p' - p)/2$, $Q = (p + q)/2$, and C is a constant containing an infrared divergence, which for scattering processes of real particles may be discarded on normalization grounds. We note that expression (4) contains no ultraviolet divergences.

Taking into account the antisymmetry of $\Gamma_{\alpha\beta,\gamma\delta}(p', q', p, q)$, we finally obtain

$$\Gamma_{\alpha\beta,\gamma\delta}(p', q', p, q) = (\sigma^n_{\alpha\beta} \times \sigma^n_{\gamma\delta})\Gamma(p', q', p, q),$$

$$\Gamma^{(2)}(p', q', p, q) = 1 - \frac{g}{\pi} \ln \frac{(p' - p)^2 (p' - q)^2}{(p + q)^4} + gc, \tag{5}$$

where c is an arbitrary finite constant.

Let us now improve the approximation properties of expression (5) by the renormalization-group method ([2]).

The renormalization group for the case under consideration has the same structure as the renormalization group for the nonlinear meson theory obtained from the two-charge meson-nucleon theory in the limit in which the meson-nucleon interaction is switched off.

The corresponding functional equations have the form ([3])

$$d(x, g) = d(t, g) d\left(\frac{x}{t}, g\varphi(t, g)\right); \tag{6}$$

$$\Gamma(x_1, \dots, x_6, g) = \Gamma(t, \dots, t, g)\Gamma\left(\frac{x_1}{t}, \dots, \frac{x_6}{t}, g\varphi(t, g)\right); \tag{7}$$

$$\varphi(t, g) = d^2(t, g)\Gamma(t, \dots, t, g). \tag{8}$$

Here d is the scalar numerator of the one-particle Green’ s function; $x_1, \dots, x_6$ are dimensionless scalar independent momentum arguments of the function Γ, which we choose in the following form:

$$x_1 = p^2/\lambda^2, \quad x_2 = {p'}^2/\lambda^2, \quad x_3 = q^2/\lambda^2, \quad x_4 = {q'}^2/\lambda^2, \tag{9}$$

$$x_5 = \frac{(p' - p)^2}{\lambda^2}, \quad x_6 = (p' - q)^2/\lambda^2,$$

where λ is the ordinary normalization momentum.

The function φ, introduced in (8), is an invariant charge. Integrating in the usual way the differential equation for φ obtained from equations (6), (7), (8), it is easy to verify that

$$\varphi(x, g) = 1. \tag{10}$$

This corresponds to the fact that in the lowest approximation $d = 1$, while the expression for Γ symmetric in the momenta according to (5) is also equal to unity. Thus, in the approximation linear in g, the charge is not renormalized, which ultimately is a consequence of the absence of an ultraviolet divergence in expression (5).

Let us proceed to the improvement of the expression for Γ in the case of scattering of two real particles, when $x_1 = x_2 = x_3 = x_4 = 0$. Denoting

$$\Gamma(0, 0, 0, 0, x, y, g) = \Gamma(x, y, g),$$

we obtain from (5)

$$\Gamma^{(2)}(x, y, g) = 1 - \frac{g}{\pi} f\left(\frac{x}{y}\right), \qquad f(z) = \ln\frac{4z}{(1+z)^2}.$$

Taking (10) into account, from equation (7) we obtain the Lie differential equation for $\Gamma(x, y, g)$:

$$\frac{\partial}{\partial x}\ln\Gamma(x, y, g) = \frac{1}{x}\frac{\partial}{\partial \xi}\ln\Gamma^{(2)}\left(\xi, \frac{y}{x}, g\right)\bigg|_{\xi=1} = -\frac{g}{\pi}\frac{\partial}{\partial x} f\left(\frac{x}{y}\right),$$

where on the right-hand side, as usual, only terms proportional to g have been retained. Integration of this equation, taking into account the property $f(1) = 1$, gives

$$\Gamma(x, y, g) = \left[\frac{4xy}{(x+y)^2}\right]^{-g/\pi}$$

or

$$\Gamma(p', q', p, q) = \left[\frac{(p'-p)^2(p'-q)^2}{(p+q)^4}\right]^{-g/\pi}. \tag{11}$$

We note that in deriving this formula only one approximation was made, namely the neglect of terms of higher order in g.

Therefore, in a certain sense, formula (11) is exact in the limit of small g, in contrast to the usual results obtained by the renormalization-group method, which are valid only in asymptotic regions of momentum variables. This fact is due to the zero mass of the particles of the ψ-field.

Let us also note that formula (11) is likewise a counterexample to the prescription for obtaining momentum asymptotics given in ([4]) (see a detailed discussion of this question in ([3])).

Formula (11) is very reminiscent of Thirring's result (formula (4.11) from ([1])), in the limit of small $g = \lambda$. Assuming that the correspondence between our result and Thirring' s result persists in higher orders in g (which is quite probable, since the models considered differ only in their symmetry properties), one may expect that, when higher approximations are taken into account, one obtains

$$\Gamma(p', q', p, q) \sim \left[\frac{(p'-p)^2(p'-q)^2}{(p+q)^4}\right]^{-\frac{1}{\pi}\operatorname{arctg} g}. \tag{12}$$

This will mean that the charge is not renormalized at all, i.e. that formula (10) remains valid in all orders in g.

The last remark is valid without any reservations for the Thirring model, which, in this way, possesses the following remarkable property. In contrast to all field theories considered so far, this model does not lead to the nonclosure of the weak-coupling method (or, equivalently, to the so-called "logarithmic-pole" difficulty).

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Note: Figure translations are in progress. See original paper for figures.