AN EXAMPLE OF A COMPLETELY REGULAR SPACE WITH A ZERO-DIMENSIONAL* ČECH REMAINDER, NOT HAVING THE PROPERTY OF SEMIBICOMPACTNESS

A. *A zero-dimensional space remains zero-dimensional after the adjunction of one point.

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Abstract

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AN EXAMPLE OF A COMPLETELY REGULAR SPACE WITH A ZERO-DIMENSIONAL* ČECH REMAINDER, NOT HAVING THE PROPERTY OF SEMIBICOMPACTNESS

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In ([1,2]) Freudenthal gave a necessary and sufficient condition in order that a space R with a countable base have a bicompact extension $\widetilde{R}$ (with a countable base) with zero-dimensional remainder $\widetilde{R} \setminus R$. This condition turned out to be the condition of **semibicompactness**: the existence at each point of the space of arbitrarily small neighborhoods with bicompact boundary. Freudenthal gave a very general proof of this interesting theorem (which, as it seemed, did not make use of the existence of a countable base) for arbitrary Hausdorff spaces. In deriving the necessity of the condition of semibicompactness he used the following assertion:

A. *A zero-dimensional space remains zero-dimensional after the adjunction of one point.*

However, this assertion, true for regular spaces with a countable base**, is, generally speaking, false.

B. **Auxiliary example.** Consider the well-known normal zero-dimensional space M of Dowker, having positive dimension defined by means of coverings***: $\operatorname{ind} M = 0$, $\dim M > 0$ (see ([3]), p. 118). In it there exist two closed disjoint sets A and B, not separated by the empty set****. Define the space P as the set consisting of all points of the space M not contained in A, and of the set A itself, with the topology customary under identification*****.

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* In the expressions "one-dimensional," "zero-dimensional," dimension is understood only in the sense of small inductive dimension, i.e. the Urysohn dimension ind, in the definition of which induction is carried out over points. The inductive dimension Ind in the sense of Čech, in the definition of which induction is carried out over closed sets, we call large, since always $\operatorname{ind} R \leq \operatorname{Ind} R$.

** Even independently of the fulfillment of the axiom of countability at the adjoined point. Indeed, if $\operatorname{ind} R = 0$, then the dimension of any regular space

$R \cup \xi$, $\xi \notin R$, at each point $x \in R$ is zero. At the same time, for any two disjoint neighborhoods $O\xi$ and $U\xi$, $U\xi \subset O\xi$, the sets $C = R \cap |U\xi|$ and $D = R \setminus O\xi$ are closed and do not intersect in the regular space R with a countable base. Therefore there exists an open-and-closed set H of the space R such that $C \subset H \subset R \setminus D = O\xi \setminus \xi$. It is easy to see that the set $H \cup \xi$ is open and closed in the space $R \cup \xi$ and $\xi \in H \cup \xi \subset O\xi$. Hence the dimension of the space $R \cup \xi$ at the point ξ is also zero.

*** Beginning here, alongside the small inductive dimension ind, we shall also consider the dimension dim defined by means of coverings. It is important to keep in mind that for normal spaces the inequalities $\operatorname{Ind} R > 0$ and $\dim R > 0$ are equivalent.

**** It is said that the sets A and B of the space R are separated by the set C if the difference $R \setminus C$ is the sum of such open sets H and G that $A \subset H$ and $B \subset G$.

***** The open sets in P are considered to be all open sets of the space M either containing the set A, or not intersecting A at all.

The space P is normal, and $\operatorname{ind}(P \setminus A) = 0$, but $\operatorname{ind} P > 0$ (since the point A cannot be separated in P from the closed set B by the empty set).

C. **Construction of the basic space W.** Take an arbitrary bicompact extension αP of the space P^*. Let $\aleph_\tau$ be the cardinality of the extension αP, and let $T_{\tau+1}$ be the bicompact space of all ordinal numbers not exceeding the first ordinal number $\omega_{\tau+1}$ of cardinality $\aleph_{\tau+1}$. Let $U_{\tau+1} = T_{\tau+1} \setminus \omega_{\tau+1}$. We shall agree to identify points and sets of the space αP with the corresponding points and sets of the product $\alpha P \times \omega_{\tau+1}$, for example the point A with the point $A \times \omega_{\tau+1}$, or the set $P \setminus A$ with the set $(P \setminus A) \times \omega_{\tau+1}$.

The required completely regular space W is defined by the formula

$$W = (\alpha P \times T_{\tau+1}) \setminus (P \setminus A) = (\alpha P \times U_{\tau+1}) \cup (\alpha P \setminus P \cup A).$$

D. **Every continuous function defined on the product $\alpha P \times U_{\tau+1}$ can be extended continuously to the bicompactum $\alpha P \times T_{\tau+1}$ ((4), p. 300).**

E. **The bicompactum $\alpha P \times T_{\tau+1}$ is a Čech extension βW of the space W, and** $\operatorname{ind}(\beta W \setminus W) = 0$.

Indeed, by D, the bicompactum $\alpha P \times T_{\tau+1}$ is a Čech extension of the product $\alpha P \times U_{\tau+1}$, and moreover

$$\alpha P \times U_{\tau+1} \subset W \subset \alpha P \times T_{\tau+1}.$$

Hence $\beta W = \alpha P \times T_{\tau+1}$. Since $\beta W \setminus W = P \setminus A$, we have $\operatorname{ind}(\beta W \setminus W) = 0$.

It remains to prove that the space W is not semibicompact. For this we need the following two assertions.

F. **In every Hausdorff semibicompact space, any two sets whose closures are separated by a bicompact set are functionally separated.**

Indeed, every such space R becomes a proximity space if in it one declares distant exactly those pairs of sets whose closures are separated by bicompact sets. All the axioms of V. A. Efremovich ([5]) are then satisfied ([6]). But then any two distant sets are functionally separated ([5]), p. 196), which is what was required to prove.

It follows from this, in particular, that every Hausdorff semibicompact space is completely regular**.

G. **Let R be a Hausdorff semibicompact space; then any bicompact set Φ which splits the space R so that $R \setminus \Phi = U \cup V$, $U \cap V = \Lambda$, also splits the Čech extension βR of the space R, and moreover so that $\beta R \setminus \Phi = U^* \cup V^*$, $U^* \cap V^* = \Lambda$, $R \cap U^* = U$ and $R \cap V^* = V$.**

Proof. First, assuming the condition of Lemma G fulfilled, we prove that

$$\Phi = \beta R[U \cup \Phi] \cap \beta R[V \cup \Phi].$$

For this, supposing the contrary, take a point ξ such that $\xi \notin \Phi$, but $\xi \in \beta R[U \cup \Phi] \cap \beta R[V \cup \Phi]$. Since the bicompactum Φ is closed in βR, there exists a neighborhood $O\Phi$ such that $\xi \notin \beta R[O\Phi]$. The closed sets

* For example, the product of the interval $[0;1]$ by the space T_1 of all ordinal numbers not exceeding the first countable ordinal ω_1 contains (in this product, according to Dowker' s construction) a space M, with $A = M \cap (0 \times T_1)$, and $B = M \cap (1 \times T_1)$, with the proper identification of all points of the "lower base" $0 \times T_1$, containing the set A, into one point, which is naturally identified with the point A of the space P.

** Lemma F was implicitly proved by Freudenthal ([1]), §§ 4.3 and 4.4); complete regularity follows directly from the main part (sufficiency) of Freudenthal' s general theorem that is of interest to us.

$P = (U \cup \Phi) \setminus O\Phi = U \setminus O\Phi$ and $Q = (V \cup \Phi) \setminus O\Phi = V \setminus O\Phi$ are separated by the bicompactum Φ. Hence, by item F, they are also functionally separated, and therefore $\beta R[P] \cap \beta R[Q] = \Lambda$. Since $U \cup \Phi \subseteq P \cup O\Phi$, we have $\beta R[U \cup \Phi] \subseteq \beta R[P] \cup \beta R[O\Phi]$.

Similarly, $\beta R[V \cup \Phi] \subseteq \beta R[Q] \cup \beta R[O\Phi]$. Hence, $\xi \in \beta R[U \cup \Phi] \cap \beta R[V \cup \Phi] \subseteq \beta R[O\Phi]$, contrary to the choice of the neighborhood $O\Phi$. Thus,

$$\Phi = \beta R[U \cup \Phi] \cap \beta R[V \cup \Phi].$$

Finally, since $(U \cup \Phi) \cup (V \cup \Phi) = R$, we also have $\beta R[U \cup \Phi] \cup \beta R[V \cup \Phi] = \beta R$. Hence, for the sets $U^* = \beta R \setminus \beta R[V \cup \Phi]$ and $V^* = \beta R \setminus \beta R[U \cup \Phi]$, we shall have: $\beta R \setminus \Phi = U^* \cup V^*$, $U^* \cap V^* = \Lambda$, $R \cap U^* = R \setminus \beta R[V \cup \Phi] = R \setminus R[V \cup \Phi] = U$, and, analogously, $R \cap V^* = V$, as was required to be proved.

H. The space W is not semicompact.

Proof. Suppose the contrary, and take a point A and a set B closed in the space P. Since $A \notin \beta W[B]$, there is in the extension βW a neighborhood O^* of the point A such that $\beta W[O^*] \cap \beta W[B] = \Lambda$. By semicompactness, in the space W there is also a neighborhood U of the point A, contained in O^*, which has a bicompact boundary Φ. For convenience assume that $W \setminus U = W[W \setminus W[U]]$, i.e., that U is a canonical open set, and denote by V the complement $W \setminus W[U]$.

Then, by item G, there exist open sets U^* and V^* in βW such that $W \cap U^* = U$, $W \cap V^* = V$, $\beta W \setminus \Phi = U^* \cup V^*$, and $U^* \cap V^* = \Lambda$. Hence we obtain:

In the space P, the set $P \cap U^*$ is an open-and-closed neighborhood of the point A and lies in $P \setminus B$.

Indeed: 1) $A \in U^*$, and therefore $A \in P \cap U^*$; 2) $\beta W[U^*] = \beta W[U] \subseteq \beta W[O^*] \subseteq \beta W \setminus \beta W[B]$, and therefore $P \cap U^* \subseteq P \setminus B$; 3) $\beta W[U^*] \setminus U^* = \beta W \setminus U^* \setminus V^*$, hence $P[P \cap U^*] \setminus (P \cap U^*) = P[P \cap U^*] \setminus U^* \subseteq \beta W[U^*] \setminus U^* = \Phi$. But $\Phi \subseteq W \setminus A = \beta W \setminus P$, and therefore the boundary of the set $P \cap U^*$ in the space P is empty, since $\Phi \cap P = \Lambda$. However, the proposition obtained contradicts the inequality $\mathrm{ind}_A\, P > 0$. Hence the space W is not semicompact, as was required to be proved.

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Note: Figure translations are in progress. See original paper for figures.