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A. E. GEL'MAN

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Abstract

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MATHEMATICS

A. E. GEL' MAN

ONE CRITERION FOR THE EXISTENCE OF CERTAIN CLASSES OF SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION AND SOME ESTIMATES IN THE METHOD OF A SMALL PARAMETER

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In the works of Krummling (^{1,2}) and Lewis (^{3}) the question is studied of estimating the domain of variation of a parameter in which there exists (or else can be expanded in a series in this parameter) a periodic solution of a nonlinear differential equation. The methods used in these works cannot, in principle, be applied to solving analogous problems for a quasi- or almost-periodic solution. The ideas developed below make it possible to solve this problem simultaneously for a periodic solution, for a quasi- or almost-periodic solution, and for a solution bounded on the whole real axis.

Lemma. Let

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_{ny},$$

where a_i are constants and the absolute values of the real parts of the roots of the characteristic polynomial of the operator are $\alpha_1, \alpha_2, \dots, \alpha_n$, with $\alpha_i \neq 0$. Let $\psi(t)$ be a bounded measurable function defined on $(-\infty, +\infty)$, and

$$\sup_{-\infty<t<+\infty} |\psi(t)| = \bar{\psi}.$$

Then there exists a unique solution, bounded on $(-\infty, +\infty)$,* of the equation

$$L(y) = \psi(t),$$

and for this solution and its derivatives the estimates** hold

$$|y^{(i)}| \leq \beta_i \bar{\psi} \qquad (0 \leq i \leq n),$$

where

$$\beta_i = u_i \frac{\alpha_1^i + \alpha_2^i + \cdots + \alpha_n^i}{n\alpha_1\alpha_2\cdots\alpha_n}; \qquad u_0 = 1; \quad u_1 = 2; \quad u_2 = 6, \dots,$$

and, in general,

$$u_i = 1 + C_i^1 u_1 + C_i^2 u_2 + \cdots + C_i^{i-1} u_{i-1} + 1.$$

Moreover, if $\psi(t)$ is a periodic, quasi- or almost-periodic function, then this bounded solution also belongs to the same classes of functions.

The assertion of the lemma is obvious for the case $n = 1$. For arbitrary n the lemma is proved by induction. (This lemma generalizes the results of Esclangon (^{4}), Bohr and Neugebauer (^{5}).)

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* By a solution in this paper is meant a function absolutely continuous together with its derivatives up to order $(n-1)$ inclusive, satisfying the given equation almost everywhere. Therefore all equalities and inequalities involving $L(y)$ or $y^{(n)}$ should be understood as holding almost everywhere.

** Of these estimates, the first is sharp:

$$|y| \leq \frac{\bar{\psi}}{\alpha_1\alpha_2\cdots\alpha_n}.$$

Definition. Let $F(t, x_0, x_1, \dots, x_n)$ be a function analytic in a neighborhood of zero with respect to $x_0, x_1, \dots, x_n$, whose Maclaurin coefficients are bounded and measurable functions of t on $(-\infty, +\infty)$. We shall call a function $\Phi(x)$ an L-majorant of $F(t, y, y', \dots, y^{(n)})$ if $\Phi(x)$ is a power series in x, obtained from the series $F(t, y, y', \dots, y^{(n)})$ by replacing all coefficients by exact upper bounds of their moduli, and $y^{(i)}$ by $\beta_i x$.

Theorem 1. Given the equation

$$L(y) = \psi(t) + F(t, y, y', \dots, y^{(n)}), \tag{1}$$

where the operator $L(y)$ and the function $\psi(t)$ are introduced in the lemma, and the function $F(t, y, y', \dots, y^{(n)})$ satisfies the conditions stated above and, moreover, $F(t, 0, 0, \dots, 0) = 0$.

Let $\Phi(x)$ be an L-majorant for $F(t, y, y', \dots, y^{(n)})$, and suppose the equation

$$\Phi'(x) + \frac{\overline{\psi}}{x} = 1 \tag{2}$$

has positive roots, and let Y be the least of them.

Then equation (1) has a particular solution, bounded on the entire real axis,

$$|y| \leqslant \frac{Y}{a_1 a_2 \dots a_n}.$$

If, in addition, $\psi(t)$ and the coefficients of the series $F(t, y, y', \dots, y^{(n)})$ are functions periodic with a common period, then this bounded solution will be periodic; if $\psi(t)$ and the coefficients of the series $F(t, y, y', \dots, y^{(n)})$ are quasiperiodic functions with a common frequency basis, then this bounded solution will be quasiperiodic with the same frequency basis; if, finally, $\psi(t)$ and the coefficients of the series $F(t, y, y', \dots, y^{(n)})$ are almost-periodic functions, then the bounded solution will also be an almost-periodic function.

The assertion of the theorem is a consequence of the Banach-Caccioppoli-Tikhonov principle, Lemma 1, and the closedness property of the classes of functions indicated in the formulation of the theorem with respect to the operation of uniform limiting passage.

Theorem 2. Given the equation

$$L(y) = \psi(\lambda, t) + F(\lambda, t, y, y', \dots, y^{(n)}), \tag{3}$$

where $F(\lambda, t, x_0, x_1, \dots, x_n)$ is a function analytic in a neighborhood of zero with respect to $\lambda, x_0, x_1, \dots, x_n$, the coefficients of whose series are bounded measurable functions of t on the interval $(-\infty, +\infty)$, and $F(\lambda, t, 0, \dots, 0) = 0$; the function $\psi(\lambda, t)$ is analytic with respect to λ in a neighborhood of zero, with coefficients of the class indicated above.

Let $\Phi(x, \lambda)$ be an L-majorant of $F(\lambda, t, y, y', \dots, y^{(n)})$; R its radius, and let $\overline{\psi}(\lambda)$ be the function obtained from $\psi(\lambda, t)$ by replacing all coefficients by exact upper bounds of their moduli.

Suppose the equation

$$\Phi'_x(x, 0) + \frac{\overline{\psi}(0)}{x} = 1$$

has a nonmultiple positive root x^0, and moreover $x_0 \neq R$.

Then there exists $\Lambda > 0$ such that for $|\lambda| < \Lambda$ there exists a solution of equation (3) possessing the properties indicated in Theorem 1*, and ana-

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* The boundedness, periodicity, quasiperiodicity, or almost-periodicity properties of the solution are meant, under the condition that the coefficients of $F(\lambda, t, y, y', \dots, y^{(n)})$ and $\psi(\lambda, t)$ belong to the corresponding classes of functions.

analytic* with respect to λ. In this case one may take $\Lambda = \sup_{0<x<R} \lambda$, where λ satisfies the equation

$$\Phi'_x(x, \lambda) + \frac{\overline{\psi}(\lambda)}{y} = 1. \tag{4}$$

If, further,

$$x(\lambda) = x_0 + \lambda x_1 + \lambda^2 x_2 + \ldots$$

is the expansion in powers of λ of the smaller positive root** of equation (4), and

$$y(\lambda, t) = y_0(t) + \lambda y_1(t) + \lambda^2 y_2(t) + \ldots \tag{5}$$

is the expansion in powers of λ of the indicated solution of equation (3), then

$$|y_k(t)| \leqslant \frac{x_k}{a_1 a_2 \ldots a_n} \qquad (-\infty < t < +\infty) * * * .$$

The proof of Theorem 2 differs little from the proof of Theorem 1.

Corollary 1. If equation (3) has the particular form

$$L(y) = \lambda\psi(t) + F(t, y, y', \ldots, y^{(n)}),$$

$\overline{\psi} \neq 0$ and $\Phi'(0) < 1$, then

$$\Lambda = \sup_{0<x<R} \frac{x(1 - \Phi'(x))}{\overline{\psi}}.$$

Corollary 2. If equation (3) has the particular form

$$L(y) = \psi(t) + \lambda F(t, y, y', \ldots, y^{(n)})$$

and $0 < \overline{\psi} < R$, then

$$\Lambda = \sup_{0<x<R} \frac{1 - \overline{\psi}/x}{\Phi'(x)}.$$

Leningrad Electrotechnical Institute
named after V. I. Ulyanov (Lenin)

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* In the literature known to the author, the fact itself of the existence of a solution analytic with respect to λ (without estimating the domain of convergence), even in the simplest periodic case, was proved only under the assumption of continuity of the coefficients $\psi(\lambda, t)$ and $F(\lambda, t, y, y', \ldots, y^{(n)})$.

** Equation (4) has no more than two positive roots.

*** From this, by the usual method, one obtains estimates for the remainder term of series (5).

Note: Figure translations are in progress. See original paper for figures.

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