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Abstract

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MATHEMATICS

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ON THE NEGATIVE PART OF THE SPECTRUM OF ONE-DIMENSIONAL AND MULTIDIMENSIONAL DIFFERENTIAL OPERATORS ON VECTOR FUNCTIONS

(Presented by Academician S. N. Bernstein, 24 X 1957)

The present note is devoted to a generalization of the theorems of note ^(1a), which supplements the results obtained earlier ^(1b) on the spectrum of one-dimensional and multidimensional differential operators on vector functions.

Let $\mathcal{L}_2(0, \infty)$ be the Hilbert space of vector functions $\mathbf{y}(x) = \{y_k\}_{k=1}^m$ ($m < \infty$) with scalar product

$$(\mathbf{y}, \mathbf{z}) = \int_0^\infty \sum_{k=1}^m y_k(x) \overline{z_k(x)} dx,$$

and let $l[\mathbf{y}]$ be a differential operation of the form

$$l[\mathbf{y}] = (-1)^n \mathbf{y}^{(2n)} + Q(x)\mathbf{y} \quad (0 \leq x < \infty), \tag{1}$$

where $Q(x)$ is a Hermitian matrix function of order m . Denote the smallest and, respectively, the largest eigenvalue of the matrix $Q(x)$ by $\mu(x)$ and $\nu(x)$. Let \tilde{L} denote any self-adjoint extension of the operator with minimal domain of definition generated in $\mathcal{L}_2(0, \infty)$ by the operation (1). The negative part of any function $f(x)$ will be denoted by $f^*(x)$, so that $f^*(x) = \min\{0, f(x)\}$.

The use of Lemma 1 of note ^(1a), where the functional $\Phi_\varepsilon[y]$ should be replaced by the functional

$$\Phi_\varepsilon[\mathbf{y}] = \int_0^\infty \sum_{k=1}^m |y_k^{(n)}(x)|^2 dx + \int_0^\infty \sum_{j,k=1}^m Q_{jk}(x) y_j(x) \overline{y_k(x)} dx + \varepsilon \int_0^\infty \sum_{k=1}^m |y_k(x)|^2 dx,$$

leads to the following results.

Theorem 1. *If for every $\delta > 0$ the inequality*

$$\int_{M_\delta} |\mu^*(x)| dx < \infty,$$

holds, where M_δ is the set of values x for which $|\mu^*(x)| \geq \delta$, then the negative part of the spectrum of the operator \tilde{L} is bounded below and discrete.

Putting, further,

$$\alpha_n = \frac{(2n-1)!!}{2^n}, \quad A_n = (2n-1)^{-1/2} \left[\sum_{k=1}^n \frac{(-1)^{k-1} C_{n-1}^{k-1}}{2n-k} \right]^{-1} (n-1)!,$$

$$B_n^2 = \frac{n(4n^2-1)}{3 \cdot 4^{n-1}} \sum_{k=1}^n \frac{1}{2k-1} \sum_{k=0}^{2n-2} \frac{(-1)^k C_{2n-2}^k}{4n-3-k} \left[\sum_{k=1}^n \frac{(-1)^{k-1} C_{n-1}^{k-1}}{2n-k} \right]^{-2},$$

we note the following two theorems.

Theorem 2. *The negative part of the spectrum of the operator \tilde{L} consists of a finite number of eigenvalues if one of the following conditions is satisfied:*

- 1°. $\mu(x) \geq -\alpha_n^2 x^{-2n}$ for large x .
- 2°. For every $\delta > 0$

$$\int_{M_\delta} x^{2n-1} |\mu^*(x)| dx < \infty,$$

where M_δ is the set of values x for which

$$|\mu^*(x)| \geq (\alpha_n^2 - \delta)x^{-2n}.$$

- 3°. For some $p \geq 1$

$$\int_0^\infty x^{2np-1} |\mu^*(x)|^p dx < \infty.$$

Theorem 3. *The negative part of the spectrum of the operator \tilde{L} is an infinite set if one of the following conditions is satisfied:*

- 1°. For some $\delta > 0$ and large x

$$\nu(x) < -(\alpha_n^2 + \delta)x^{-2n}.$$

- 2°. $\nu(x) \leq 0$ for large values of x and

$$\liminf_{\rho \rightarrow \infty} \rho^{2n-1} \int_{\rho}^{\infty} |\nu(x)| dx > A_n^2.$$

3°. $\nu(x) \leq -\alpha_n^2 x^{-2n}$ for large x and

$$\liminf_{\rho \rightarrow \infty} \ln \rho \int_{\rho}^{\infty} x^{2n-1} |\nu(x) + \alpha_n^2 x^{-2n}| dx > B_n^2.$$

4°.

$$\int_0^{\infty} \nu(x) dx = -\infty.$$

In conditions 2° and 3°, one may replace $\liminf_{\rho \rightarrow \infty}$ by $\lim_{\rho_k \rightarrow \infty}$.

Theorems 1-3 are connected with the oscillatory properties of the system of differential equations

$$(-1)^n y^{(2n)} + Q(x)y = \lambda y \quad (\lambda \leq 0),$$

which, for $n = 1$, were studied by Sternberg ⁽²⁾. Conditions 1° of Theorem 2 and 1° of Theorem 3 give a generalization of the well-known Kneser theorem on oscillation.

solutions of a second-order differential equation. For $n = 1$ one can obtain the following refinement of condition 1° of Theorem 3, which for the case of a second-order differential equation was given by Hill ⁽³⁾ (see also ⁽⁴⁾).

Theorem 4. If, for some $\delta > 0$ and some natural number r , for all sufficiently large values of x the inequality

$$\nu(x) < -\frac{1}{4x^2} - \frac{1}{4x^2 \ln^2 x} - \dots - \frac{1 + \delta}{4x^2 \ln^2 x \dots \ln_r^2 x},$$

holds, where $\ln_k x = \ln \ln_{k-1} x$, then the negative part of the spectrum of the operator \tilde{L} consists of a finite number of eigenvalues.

The results presented extend in part to multidimensional differential operations on vector-functions of the form

$$l[u] = -\Delta u + Q(P)u, \tag{2}$$

where P is a point of n -dimensional Euclidean space \mathcal{E} ; $Q(P)$ is a Hermitian matrix-function of order m , defined on all of \mathcal{E} .

The operation (2) generates in the Hilbert space $\mathfrak{L}_2(\mathcal{E})$ of vector-functions $\mathbf{u}(P) = \{u_k(P)\}_{k=1}^m$, with scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{E}} \sum_{k=1}^m u_k(P) \overline{v_k(P)} d\omega_P,$$

a certain differential operator L with minimal domain of definition.

Let $\mu(P)$ be the least eigenvalue of the matrix $Q(P)$, and let $\mu^*(P) = \min\{0, \mu(P)\}$. We give, for example, the statement of the theorem corresponding to Theorem 1, and prove it for $m = 1$, $n = 2$ (in this case $\mu(P) = Q(P)$).

Theorem 5. If, for every $\delta > 0$, the integral

$$\int_0^\infty |\mu_\delta^*(P)| dr,$$

where

$$\mu_\delta^*(P) = \begin{cases} \mu^*(P), & |\mu^*(P)| > \delta, \\ 0, & |\mu^*(P)| \leq \delta, \end{cases}$$

converges uniformly with respect to the angular coordinates, then:

- 1) the operator L with minimal domain of definition (see (1^B)) is self-adjoint;
- 2) the negative part of the spectrum of the operator L is bounded below and discrete (i.e. consists of eigenvalues of finite multiplicity with the only possible limit point $\lambda = 0$).

Proof. Transforming the quadratic functional

$$\Phi_\varepsilon[u] = \iint_{(\mathcal{E})} |\nabla u|^2 r dr d\varphi + \iint_{(\mathcal{E})} Q|u|^2 r dr d\varphi + \varepsilon \iint_{(\mathcal{E})} |u|^2 r dr d\varphi$$

on any finite function $u \in D_L$ by the change of variables $u\sqrt{r} = v$, we obtain

$$\Phi_\varepsilon[u] = \iint_{(\mathcal{E})} |\nabla v|^2 dr d\varphi + \iint_{(\mathcal{E})} \left[Q(r, \varphi) + \frac{1}{4r^2} \right] |v|^2 dr d\varphi + \varepsilon \iint_{(\mathcal{E})} |v|^2 dr d\varphi.$$

For an arbitrarily given ε ($0 < \varepsilon < 1$), choose the number N so that

$$\int_N^\infty |Q^*(r, \varphi)| dr < \frac{\varepsilon}{4},$$

and show that the functional

$$\Phi_\varepsilon[u] = \int_0^{2\pi} d\varphi \int_N^\infty \left\{ |\nabla v|^2 + \left[Q^*(r, \varphi) + \frac{1}{4r^2} + \varepsilon \right] |v|^2 \right\} dr$$

is nonnegative on any finite function $v \in D_L$ equal to zero in the disk $r < N$.

From the Cauchy–Bunyakovsky inequality it follows that

$$\int_0^{2\pi} \int_N^\infty |\nabla v|^2 dr d\varphi \geq \int_0^{2\pi} d\varphi \int_N^\infty \left| \frac{dv}{dr} \right|^2 dr \geq \frac{1}{4} \left\{ \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \right\}^2,$$

where the function $v(r, \varphi)$ is normalized by the condition

$$\int_0^{2\pi} d\varphi \int_N^\infty |v|^2 dr = 1$$

and

$$\hat{v}(\varphi) = \max_{N < r < \infty} |v(r, \varphi)|.$$

Consider separately two cases:

$$1^\circ. \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \leq \varepsilon. \quad 2^\circ. \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi > \varepsilon.$$

In the first case

$$\begin{aligned} \Phi_\varepsilon[u] &\geq \int_0^{2\pi} \int_N^\infty Q^*(r, \varphi) |v|^2 dr d\varphi + \varepsilon \int_0^{2\pi} \int_N^\infty |v|^2 dr d\varphi \geq \\ &\geq - \int_0^{2\pi} |\hat{v}(\varphi)|^2 \int_N^\infty |Q^*(r, \varphi)| dr d\varphi + \varepsilon \geq \varepsilon \left(1 - \frac{\varepsilon}{4} \right) > 0. \end{aligned}$$

In the second case

$$\Phi_\varepsilon[u] \geq \frac{1}{4} \left\{ \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \right\}^2 + \int_0^{2\pi} \int_N^\infty Q^*(r, \varphi) |\hat{v}(\varphi)|^2 dr d\varphi,$$

so that

$$\Phi_\varepsilon[u] \geq \frac{1}{4} \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \left[\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi - \varepsilon \right] > 0,$$

and the inequality $\Phi_\varepsilon[u] \geq 0$ is established.

From this inequality, first of all, follows (1^Γ) the boundedness from below of the operator L , and hence, by a theorem of A. Ya. Povzner (⁵), the self-adjointness of the operator L follows.

Next, from this same inequality, on the basis of (1^Γ) and Lemma 1 (1^a), we conclude that the negative part of the spectrum of the operator L is discrete. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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