HOMOLOGICAL PROPERTIES OF INVERSE IMAGES OF POINTS UNDER DIMENSION-RAISING MAPPINGS OF MANIFOLDS

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Abstract

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HOMOLOGICAL PROPERTIES OF INVERSE IMAGES OF POINTS UNDER DIMENSION-RAISING MAPPINGS OF MANIFOLDS

(Presented by Academician P. S. Aleksandrov, 12 V 1958)

§ 1. In the study of dimension-raising mappings, the Vietoris theorem (1) is useful in many cases. Begle (2,3) extended this theorem from the case of compact metric spaces to the case of arbitrary bicompact Hausdorff spaces. We shall need the Vietoris theorem in the following formulation (2,3):

If, under a continuous mapping f of a space X onto a space Y, the complete inverse image of each point of the space Y is homologically trivial in dimensions $\leq n$, then the homomorphism of the k-dimensional homology group $H_k(X)$ of the space X into the group $H_k(Y)$ of the space Y, induced by the mapping f, is an isomorphism onto, $k \leq n$, and the homomorphism of the group $H_{n+1}(X)$ into $H_{n+1}(Y)$ is a homomorphism onto the whole group $H_{n+1}(Y)$; here the coefficient group is a field or an elementary compact group (the character group of a discrete group with a finite basis).

Dyer (4) showed that the following theorem follows immediately from the Vietoris theorem:

If, under a mapping of a compact metric space M onto a compact metric space N, the inverse image of each point y of N is acyclic in all dimensions, i.e. $H_k(f^{-1}(y)) = 0$ for all $k \geq 0$, then $\dim M \geq \dim N$.

L. V. Keldysh (5), for arbitrary $n \geq 3$ and $k \geq 1$, constructed an example of a monotone mapping of the n-dimensional cube E^n onto the $(n+k)$-dimensional cube E^{n+k}.

L. V. Keldysh indicated that, using the properties of the mapping of E^3 onto E^4 given in that work, it is easy to construct an example of such a monotone mapping of the three-dimensional sphere S^3 onto S^4 that in every neighborhood in S^4 there are two points whose inverse images contain linked cycles.

P. S. Aleksandrov proposed considering the question of the homological properties of inverse images of various sets under dimension-raising mappings of manifolds.

In this note it is proved that it is impossible to map an n-dimensional closed orientable manifold M^n onto a polyhedron K of larger dimension in such a way that the inverse images of all points of K are acyclic in all dimensions $\leq [\frac{n-1}{2}]$. The question of the linking of inverse images of points under mappings of a three-dimensional manifold with increase of dimension is considered.

By M^n we shall denote a closed orientable n-dimensional manifold; by $p^s(M^n)$, the rank of the s-dimensional homology group of M^n.

As the coefficient group we shall take the field of rational numbers or J_m, the group of residues modulo m, for which the Vietoris theorem is valid in the formulation given above.

§ 2. **Theorem 1.** *Let f be a continuous mapping of M^n onto an m-dimensional polyhedron K, $m > n$, and let the inverse images of all points of K be acyclic in all dimensions $\leq s$. Then $2s < n - 2$.*

Proof. Suppose the contrary, i.e. $2s \geq n - 2$. Take an m-dimensional ball U^m in K, and let S^q and S^p be two such spheres lying in U^m that their fundamental cycles z^p and z^q are linked, $p + q = m - 1$. Take $p = s + 1$. By the Vietoris theorem, in the set $f^{-1}(S^p)$ there exists a p-dimensional cycle ζ^p such that $f(\zeta^p) \sim z^p$ on S^p, i.e. $f(\zeta^p) = z^p$.

Since $f(\zeta^p) = z^p \nsim 0$ in $K \setminus S^q$, it follows that $\zeta^p \nsim 0$ in $M^n \setminus f^{-1}(S^q)$. Therefore the set $f^{-1}(S^q)$ contains an r-dimensional cycle z^r linked with ζ^p, where $r = n - p - 1$. Since $p = s + 1$, we have

$$r = n - p - 1 = n - (s + 1) - 1 = n - s - 2.$$

Hence, taking into account that $2s \geq n - 2$, we obtain $r = n - s - 2 \leq s$.

From the equalities $p + q = m - 1$, $p + r = n - 1$, and the inequality $m > n$, it follows that $r < q$. Since $r \leq s$, the Vietoris theorem is applicable to the sets S^q and $f^{-1}(S^q)$. We obtain

$$H_r(f^{-1}(S^q)) = H_r(S^q) = 0,$$

since $r < q$. But this contradicts the fact that the cycle $z^r \nsim 0$ on $f^{-1}(S^q)$. Thus the assumption $2s \geq n - 2$ is false. The theorem is proved.

§ 3. In this section f denotes a monotone mapping, i.e. such a mapping under which the inverse image of a connected set is connected. It follows that if f is a monotone mapping of M^n onto a polyhedron K, then, in order that the inverse image of a compact set F from K separate M^n, it is necessary and sufficient that F separate K.

Theorem 2. *Let f be a monotone mapping of M^3 onto M^m, $m > 3$. In M^m there exist at most $p^1(M^3)$ such two-dimensional polyhedra each of which is an essential carrier of a two-dimensional cycle and, for every point y of these polyhedra,*

$$H_1(f^{-1}(y)) = 0.$$

Proof. Let K^2 be a two-dimensional polyhedron in M^m which is an essential carrier of a cycle z^2, and suppose that for every point $y \in K^2$

$$H_1(f^{-1}(y)) = 0.$$

By the Vietoris theorem, in the set $F = f^{-1}(K^2)$ there is a cycle ζ^2 such that $f(\zeta^2) \sim z^2$ on K^2. Since $f(\zeta^2) \nsim 0$ on $f^{-1}(F) = K^2$, it follows that $\zeta^2 \nsim 0$ on F. If $\zeta^2 \sim 0$ on M^3, then F separates M^3, which cannot be, since K^2 does not separate M^m and $F = f^{-1}(K^2)$. Thus $\zeta^2 \nsim 0$ on M^3.

If one assumes that the theorem is false, then there exist r such two-dimensional polyhedra $K_1^2, K_2^2, \dots, K_r^2$, $r > p^1(M^3)$, such that the set $F_i = f^{-1}(K_i^2)$ is a carrier of a cycle ζ_i^2, $\zeta_i^2 \nsim 0$ on M^3, $1 \leq i \leq r$. The cycles $\zeta_1^2, \zeta_2^2, \dots, \zeta_r^2$ are dependent, since $r > p^1(M^3)$, and therefore the compact set

$$B = \bigcup_{i=1}^{r} F_i$$

is an essential carrier of a cycle ζ^2, and $\zeta^2 \sim 0$ on M^3. This contradicts the fact that $B = f^{-1}(K)$, where

$$K = \bigcup_{i=1}^{r} K_i^2,$$

and K does not separate M^m.

Theorem 3. *Let f be a monotone mapping of M^3 onto M^m, $m > 3$, and let a be an arbitrary point in M^m. If $f^{-1}(a)$ is not a carrier of a one-dimensional cycle not homologous to zero on M^3, then in every neighborhood of a there are two points whose inverse images contain linked cycles.*

Proof. Let V be an arbitrary neighborhood of a. By the continuity of f there exists a neighborhood U of the point a such that, for every point $x \in U$, the set $f^{-1}(x)$ contains no one-dimensional cycles not homologous to zero in M^3.

By Theorem 2, in a sufficiently small neighborhood of a every two-dimensional sphere contains such a point y that $H_1(f^{-1}(y)) \neq 0$. We shall assume that U also satisfies this condition. Take $\varepsilon > 0$ such that $O(a, \varepsilon) \subset V \cap U$. In $O(a, \varepsilon)$ there is a point p such that $F = f^{-1}(p)$ contains a cycle ζ^1, $\zeta^1 \nsim 0 \cdots$

on F; $\zeta^1 \sim 0$ in M^3, since $p \in U$. Let z^1 be a polyhedral cycle in M^3, linked with ζ^1, lying outside F; K a polyhedron that is the body of z^1. The cycle z^1 can be chosen so that the cycle $f(z^1)$ lies in $O(a, \varepsilon)$ and is homologous to zero in $O(a, \varepsilon)$. For this it is necessary to take an arbitrary cycle u^1, linked with ζ^1, and a chain w^2, $\Delta w^2 = u^1$. The polyhedral neighborhood $O(F, \beta)$ cuts from the chain w^2 a chain w_1^2; Δw_1^2 is the desired cycle, if β is sufficiently small.

Let $B = f(K)$; $2\alpha = \rho(p, B)$ (ρ is the distance between sets); $G = O(a, \varepsilon) \setminus O(p, \alpha)$; Q is a triangulation of K. The images of the simplices of Q have diameter $\leqslant \delta(Q)$; one may always assume that $\delta(Q) < \alpha$. We shall prove that in G there exists a ball $W(Q)$ of radius $\leqslant \delta(Q)$, whose preimage contains a cycle not homologous to zero outside F. We have

$$z^1 = \sum_{i=1}^{s} b_i t_i^1, \qquad \Delta t_i^1 = h_i - g_i.$$

Let

$$c_i = f(h_i); \qquad d_i = f(g_i);$$

l_i be a segment in M^m joining c_i with d_i. By virtue of the monotonicity of f, in the set $f^{-1}(l_i)$ there exists a chain x_i^1 such that

$$\Delta x_i^1 = h_i - g_i.$$

The sets $f(\widetilde{t_i^1})$ and l_i lie in a ball of radius $\delta(Q)$ with center at c_i, and the point p is at a distance greater than α from this ball. If, for some i, $1 \leqslant i \leqslant s$, the cycle $(t_i^1 - x_i^1) \nsim 0$ in $M^3 \setminus F$, then the ball $W(Q)$ exists. Consider the case when for all i, $1 \leqslant i \leqslant s$, $(t_i^1 - x_i^1) \sim 0$ in $M^3 \setminus F$.

Then the cycle

$$y^1 = \sum_{i=1}^{s} b_i x_i^1 \sim \sum b_i t_i^1 = z^1,$$

i.e. y^1 is linked with ζ^1 and lies outside F.

Let x^2 be a chain in G, bounding the cycle $f(y^1)$, and whose simplices have diameter $< \delta(Q)$,

$$x^2 = \sum_{i=1}^{n} e_i \tau_i^2.$$

In the set $f^{-1}(\widetilde{\Delta\tau_i^2})$ we choose a cycle η_i^1 such that

$$f(\eta_i^1) = \Delta\tau_i^2,$$

and the cycle η_i^1 is formed by chains defined in the same way as the chains x_i^1 introduced above. Therefore

$$\sum_{i=1}^{n} e_i \eta_i^1 = y^1.$$

We shall prove that there exists a k, $1 \leqslant k \leqslant n$, such that $\eta_k^1 \nsim 0$ in $M^3 \setminus F$. In the contrary case we have a chain v_i^2, $\Delta v_i^2 = \eta_i^1$, and v_i^2 lies in $M^3 \setminus F$. The chain

$$\sum_{i=1}^{n} e_i v_i^2$$

lies outside F and bounds the cycle y^1, which is linked with ζ^1, while ζ^1 lies in F. The ball containing the image of the cycle η_k^1 has diameter $< \delta(Q)$ and, consequently, is the desired ball.

Take a sequence of refining triangulations Q_j, $\delta(Q_j) \to 0$, and choose from it such a subsequence j_k that the balls $W(Q_{j_k})$ converge to a point q. It is clear that $q \in G$, i.e. $\rho(p, q) > \alpha$, and $f^{-1}(q)$ contains a cycle $z_0^1 \nsim 0$ outside F, since such a cycle is contained in $f^{-1}(W(Q_{j_k}))$, $k = 1, 2, \ldots$. Since $q \in U$, $z_0^1 \sim 0$ in M^3, and therefore $F = f^{-1}(p)$ contains a cycle linked with z_0^1. The points p and q lie in V. The theorem is proved.

Remark. The theorem remains valid if f is a monotone mapping of a three-dimensional manifold onto a dimensionally homogeneous polyhedron of higher dimension.

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Note: Figure translations are in progress. See original paper for figures.

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