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Abstract

Full Text

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ON IMPLICIT OPERATORS

(Presented by Academician I. G. Petrovskii, 11 I 1958)

In the present note the domain of convergence of the power series of an implicit operator is considered, estimates are given for the remainder term of this series, and possibilities are indicated for applying the results obtained to the approximate solution of functional equations.

Let X, Y , and Z be Banach spaces, and let $F(x, y)$ be an operator from the direct sum $X \dot{+} Y$ into Z . Considering $X \dot{+} Y$ as a new Banach space, one may use the concept of analyticity of the operator $F(x, y)$ given in ⁽¹⁾. It is not difficult to show that analyticity, in the sense of ⁽¹⁾, of the operator $F(x, y)$ at the point $(x_0, y_0) \in X \dot{+} Y$ is equivalent to the existence at this point of partial derivatives of all orders ^(2, 3) and to the representability of the operator $F(x, y)$ in some neighborhood of the point (x_0, y_0) by the convergent series

$$F(x, y) = \sum_{i, k=0}^{\infty} \frac{1}{i! k!} F_{x^i y^k}(x_0, y_0)(y - y_0)^k (x - x_0)^i.$$

Suppose that:

1° $F(x_0, y_0) = 0$.

2° There exists a continuous operator

$$\Gamma_0 = [F_x(x_0, y_0)]^{-1}.$$

3° $F(x, y)$ is analytic at the point (x_0, y_0) .

Then, for an arbitrary fixed $y' \in Y$, there exist nonnegative constants η, a , and a_{ik} such that

$$\begin{aligned} \|\Gamma_0 F_y(x_0, y_0)(y' - y_0)\| &\leq a\eta, \\ \|\Gamma_0 F_{x^i y^k}(x_0, y_0)(y' - y_0)^k\| &\leq a_{ik}\eta^k \quad (i, k = 0, 1, \dots, i + k \geq 2) \end{aligned} \quad (1)$$

and the power series

$$g(\alpha, \beta) = a\beta + \sum_{\substack{i, k=0 \\ i+k \geq 2}}^{\infty} \frac{1}{i! k!} a_{ik} \alpha^i \beta^k$$

converges in some neighborhood of the point $(0, 0)$. On the basis of the implicit-function theorem, the equation $\alpha = g(\alpha, \beta)$ determines an analytic function

$$\alpha = \varphi(\beta) = \sum_{k=1}^{\infty} \frac{1}{k!} \varphi^{(k)}(0) \beta^k, \quad (2)$$

which satisfies the condition $\varphi(0) = 0$.

Theorem 1. *If conditions 1°-3° are satisfied, then in a neighborhood of the point y_0 there exists an analytic operator $x = \Phi(y)$, defined by the equation*

$$F(x, y) = 0 \quad (3)$$

and the condition $\Phi(y_0) = x_0$, moreover

$$\|\Phi^{(k)}(y_0)(y' - y_0)^k\| \leq \varphi^{(k)}(0) \eta^k \quad (k = 1, 2, \dots).$$

The proof of this theorem can be carried out analogously to the proof of Theorem 1 in Note (4).

On the convergence of the power series of the implicit operator

$$\Phi(y') = x_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \Phi^{(k)}(y_0)(y' - y_0)^k \quad (4)$$

one can prove the following theorem.

Theorem 2. Suppose that conditions 1° and 2° are satisfied, that the power series (2) converges for $\beta = \eta$, and that $F(x, y)$ is analytic on the set

$$\|x - x_0\| \leq \varphi(\eta), \quad y = y_0 + t(y' - y_0) \quad (0 \leq t \leq 1). \quad (5)$$

Then equation (3), for fixed $y = y'$, has in the sphere (5) a solution x' , to which the power series (4) converges with the rate

$$\|x' - x_n\| \leq \varphi(\eta) - \sum_{k=1}^n \frac{1}{k!} \varphi^{(k)}(0) \eta^k \quad (n = 1, 2, \dots), \quad (6)$$

where

$$x_n = x_0 + \sum_{k=1}^n \frac{1}{k!} \Phi^{(k)}(y_0)(y' - y_0)^k.$$

The proof of this theorem is based on Theorem 1 and is analogous to the proof of Theorem 2 in (4).

Remark 1. If the conditions of Theorem 2 are strengthened by requiring analyticity of the function $\varphi(\beta)$ at $\beta = \eta$, then it can be proved that equation (3) for $y = y'$ has in the sphere (5) no other solutions besides x' .

Remark 2. For all operators satisfying conditions 2° and 3°, in the estimates (1) one may choose

$$a_{ik} = (i+k)!hb^{i+k-2}c^k.$$

In this case

$$\varphi(\beta) = \frac{1}{2(b+h)} \left[1 + (a-c)b\beta - 2hc\beta - \sqrt{[1 - (a+c)b\beta]^2 - 4h(a+c)\beta} \right],$$

$$\varphi'(0) = a,$$

$$\frac{1}{k!}\varphi^{(k)}(0) = \sum_{i=0}^{k-2} \frac{1}{i+1} \binom{k-2}{i} \binom{k+i}{i} b^{k-i-2} h^{i+1} (a+c)^k \quad (k = 2, 3, \dots),$$

moreover the series (2) converges for $\beta = \eta$ when

$$(a+c)b\eta + 2\sqrt{h(a+c)\eta} \leq 1.$$

If $m = 2(2h+b)(a+c) < 1$, then from (6) there follows the simpler estimate

$$\|x' - x_n\| < \frac{h(a+c)^2\eta^2}{1-m} \frac{m^{n-1}}{n} \quad (n = 1, 2, \dots).$$

Thus, Theorem 2 takes a form of which, as a special case (for $F(x; y) = P(x) - y$), Theorem 2 from (4) is a somewhat improved version.

By the method of expansion into the series of an implicit operator one can also interpret the perturbation method (see, for example, (5)). Therefore Theorem 2 gives

conditions for the convergence of the perturbation method. We shall show what results can be obtained in one comparatively simple case.

Let A, B , and y be linear operators from the space X into Z ; let D_A, D_B , and D_y be their domains of definition, and let A^*, B^* be the operators adjoint to A and B .

Suppose that:

- a) D_A is dense in X , $D_B \supset D_A$, and $D_y \supset D_A$;
- b) λ_0 is an eigenvalue of the equation $Ax = \lambda Bx$, and $\overline{\lambda_0}$ is an eigenvalue of the equation $A^*z^* = \lambda B^*z^*$, the corresponding eigenelements x_0 and z_0^* being normalized by the condition $z_0^*Bx_0 = 1$;
- c) the operator $A - \lambda B$, considered on $X_{B^*z_0^*} \cap D_A$, has a bounded inverse R , defined on $Z_{z_0^*}$;
- d) the estimates

$$\|z_0^*y\| \leq p, \quad |z_0^*yx_0| \leq p_0, \quad \|Ry\| \leq q, \quad \|Ryx_0\| \leq q_0, \quad \|RB\| \leq r$$

hold.

Under these conditions, the determination of an eigenvalue λ' of the equation

$$(A + y)x = \lambda Bx \tag{7}$$

with an eigenelement x' normalized by the condition

$$z_0^*Bx' = 1, \tag{8}$$

is equivalent to solving the equation

$$F(x, y) = x - x_0 - (z_0^*yx)RBx + Ryx = 0, \tag{9}$$

where the operator R is extended, by the equality $RBx_0 = 0$, to all of Z . In this case

$$\lambda' = \lambda_0 + z_0^*yx'.$$

In the case under consideration

$$\varphi(\eta) = \frac{1}{2pr} \left[1 - p_0r - q - \sqrt{(1 - p_0r - q)^2 - 4pq_0r} \right],$$

$$\frac{1}{k!} \varphi^{(k)}(0) \eta^k = q_0 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{i+1} \binom{2i}{i} \binom{k-1}{2i} (p_0r + q)^{k-2i-1} (pq_0r)^i \quad (k = 1, 2, \dots),$$

and theorem 3 follows from theorem 2.

Theorem 3. *If conditions a)–d) are satisfied and*

$$p_0r + q + 2\sqrt{pq_0r} \leq 1,$$

then equation (7) has an eigenvalue

$$|\lambda' - \lambda_0| \leq p\varphi(\eta) + p_0,$$

to which there corresponds an eigenelement x' , normalized by condition (8), lying in the sphere (5). The expansion of the operator $x = \Phi(y)$ (defined by equation (9)) in a power series at the point $y_0 = 0$ converges to x' with rate (6), and the sequence

$$\lambda_n = \lambda_0 + z_0^* y x_{n-1} \quad (n = 1, 2, \dots)$$

converges to λ' with rate

$$|\lambda' - \lambda_n| \leq p \left[\varphi(\eta) - \sum_{k=1}^{n-1} \frac{1}{k!} \varphi^{(k)}(0) \eta^k \right] \quad (n = 1, 2, \dots).$$

In numerical examples this theorem proved more accurate than the corresponding theorems from the works (5, 6).

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CITED LITERATURE

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$$* \quad X_{B^* z_0^*} \text{ and } Z_{z_0^*}$$

are subspaces of the spaces X and Z , satisfying respectively the conditions $B^* z_0^* x = 0$ and $z_0^* z = 0$.

Note: Figure translations are in progress. See original paper for figures.

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