FOSS SURFACES IN \(E_4\)

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Abstract

Full Text

MATHEMATICS

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FOSS SURFACES IN E_4

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1. Foss surfaces (^{1}) were defined by this author in E_3 as surfaces possessing a conjugate system consisting of two families of geodesic lines. They are mapped by parallel normals onto the pseudosphere in such a way that the conjugate Foss net passes into asymptotic lines. They admit continuous bending with preservation of the Foss system. Goursat (^{2}), considering congruences with focal nets consisting of lines of curvature, finds a Foss surface as that surface of centers of the focal surface for which the normal is parallel to the ray of the Goursat congruence. In the present note we consider two-dimensional surfaces in E_4 with a conjugate Foss net; all the properties of these remarkable surfaces carry over to them, up to a certain limit.

2. If to the points M of the surface one attaches a frame of four unit vectors e_i $(i = 1, 2)$, lying along the tangents to the lines of the Foss net, and e_α $(\alpha = 3, 4)$, orthogonal to the tangent plane and to each other, then, putting $e_1 e_2 = \cos\varphi$, for the components of the infinitesimal displacements of the frame we obtain

$$dM = \omega^i e_i, \qquad de_i = \omega_i^j e_j + \omega_i^\alpha e_\alpha, \qquad de_\alpha = \omega_\alpha^i e_i + \omega_\alpha^\beta e_\beta \tag{1}$$

the relations

$$\omega_3^4 + \omega_4^3 = 0, \qquad \omega_1^2 + \omega_2^1 + \frac{d\varphi}{\sin\varphi} = 0, \qquad \omega_i^i = -\omega_i^j \cos\varphi;$$

$$\omega_3^3 = \omega_4^4 = 0, \qquad \omega_\alpha^i \sin^2\varphi = \omega_j^\alpha \cos\varphi - \omega_i^\alpha \quad (i \neq j). \tag{2}$$

By exteriorly differentiating the equations $\omega^\alpha = 0$, after expansion by Cartan' s lemma we obtain

$$\omega_i^\alpha = l_{ij}^\alpha \omega^j.$$

Hence the two asymptotic forms Φ^α and the geodesic curvature k_g are:

$$\Phi^\alpha = l^\alpha_{ij}\omega^i\omega^j, \qquad k_g = \frac{d\alpha}{ds} + \frac{\omega_1^2}{ds}\sin\varphi, \tag{3}$$

where α is the angle made by the tangent to the line with the axis e_1.

Now all the requirements for a Foss surface will be

$$l^\alpha_{12} = 0, \qquad [\omega_1^2\omega^2] = 0, \qquad [\omega_2^1\omega^1] = 0$$

or

$$\omega_i^\alpha = l^\alpha_{ii}\omega^i, \qquad \omega_i^j = h_i\omega^j \quad \text{(do not sum! } i \neq j). \tag{4}$$

Exteriorly differentiating, we obtain

$$[\Delta l^\alpha_{ii}, \omega^i] = 0, \qquad [\omega^i h_i, \omega^j] = 0; \tag{5}$$

$$\Delta l^\alpha_{ii} = dl^\alpha_{ii} + l^\beta_{ii}\omega^\alpha_\beta + l^\alpha_{ii}(h_j + 2h_i\cos\varphi)\omega^j, \tag{6}$$

$$\Delta h_i = dh_i + \left\{(h_i)^2 + \frac{K}{\sin^2\varphi}\right\}\omega^i, \qquad K = l^3_{11}l^3_{22} + l^4_{11}l^4_{22}.$$

Since, with the 6 quadratic equations (5), each equation contains a new form $\Delta l^\alpha_{ii}, \Delta h_i$, the system is in involution and determines Foss surfaces in E_4 with arbitrary $s_1 = 6$ functions of one argument.

3. It is not hard to notice that the Foss surfaces (4), (5) admit a continuous bending on the principal base consisting of two families of geodesic lines $\omega^2 = 0$ and $\omega^1 = 0$. It is enough, while preserving the same values of the forms $\omega^1, \omega^2, \omega_1^2, \omega_2^1$, as well as φ and h_i, to assign the new quantities

$$\bar{l}^\alpha_{11} = \nu l^\alpha_{11}, \qquad \bar{l}^\alpha_{22} = \frac{1}{\nu} l^\alpha_{22}, \qquad \nu = \text{const}.$$

Then, for the new surface, we have

$$\bar{K} = K, \qquad \Delta\bar{l}^\alpha_{11} = \nu\Delta l^\alpha_{11}, \qquad \Delta\bar{l}^\alpha_{22} = \frac{1}{\nu}\Delta l^\alpha_{22}, \qquad \Delta\bar{h}_i = \Delta h_i.$$

Equations (4), (5) will be satisfied, and the linear element

$$ds^2 = (\omega^1)^2 + 2\omega^1\omega^2\cos\varphi + (\omega^2)^2$$

will be preserved. Consequently, when the parameter ν is changed, the Foss surface will be bent continuously, preserving its Foss system.

4. The congruence of tangents to the geodesics $\omega^2 = 0$ or $\omega^1 = 0$ of the surface (M) admits a family of orthogonal surfaces. If, for example, the radius vector

$$\mathbf{M}_1 = \mathbf{M} + \lambda \mathbf{e}_1 \tag{7}$$

describes, by its endpoint, a surface orthogonal to the rays $\mathbf{e}_1$, then we have $d\mathbf{M}_1 \cdot \mathbf{e}_1 = 0$, or, differentiating (7),

$$d\lambda + \omega^1 + \omega^2 \cos\varphi = 0. \tag{8}$$

This equation is completely integrable and determines a family of orthogonal surfaces (M_1), since the exterior differential, by virtue of (2), vanishes. Differentiating (7), we have

$$d\mathbf{M}_1 = \mathbf{i}_1 \lambda H_1 \omega^1 + \mathbf{i}_2 (1 + \lambda h_1) \sin\varphi\, \omega^2, \tag{9}$$

where

$$\mathbf{i}_i = \frac{l^3_{11}\mathbf{e}_3 + l^4_{11}\mathbf{e}_4}{H_1}, \qquad \mathbf{i}_2 = \frac{\mathbf{e}_2 - \mathbf{e}_1 \cos\varphi}{\sin\varphi}, \qquad H_i^2 = (l^3_{ii})^2 + (l^4_{ii})^2 \tag{10}$$

are unit orthogonal vectors.

Differentiating once more, we obtain

$$\begin{aligned} d\mathbf{i}_1 &= -\mathbf{i}_2 \frac{K}{H_1 \sin\varphi}\,\omega^2 - \left\{ H_1 \frac{\mathbf{e}_1 - \mathbf{e}_2 \cos\varphi}{\sin^2\varphi} + \frac{l^{[4}_{11} l^{3]}_{111}}{H_1^2} \frac{l^3_{11}\mathbf{e}_4 - l^4_{11}\mathbf{e}_3}{H_1} \right\} \omega^1, \\ d\mathbf{i}_2 &= -\mathbf{i}_1 H_1 \operatorname{ctg}\varphi\, \omega^1 + \left\{ -h_1 \sin\varphi\, \mathbf{e}_1 + \frac{H_2}{\sin\varphi} \frac{l^3_{22}\mathbf{e}_3 + l^4_{22}\mathbf{e}_4}{H_2} \right\} \omega^2. \end{aligned} \tag{11}$$

It is not difficult to show that the net of lines corresponding to the Foss net on the surface (M_1) is not only orthogonal, but also conjugate; namely, it serves as the focal net for the congruences $(M_1, \mathbf{i}_1)$, $(M_1, \mathbf{i}_2)$. If, for example, $\mathbf{M}_2 = \mathbf{M}_1 + \rho \mathbf{i}_1$ is the focus of the congruence $(M_1, \mathbf{i}_1)$, then, for a suitable displacement,

$$d\mathbf{M}_2 = \theta \mathbf{i}_1, \tag{12}$$

but

$$d\mathbf{M}_2 = \left\{ \mathbf{i}_1(\lambda H_1 + \rho_1) + \rho \frac{H_1}{\sin\varphi} \frac{\mathbf{e}_1 - \mathbf{e}_2 \cos\varphi}{\sin\varphi} + \rho \frac{l^{[4}_{11} l^{3]}_{111}}{H_1^2} \frac{l^3_{11}\mathbf{e}_4 - l^4_{11}\mathbf{e}_3}{H_1} \right\} \omega^1 + \left\{ \mathbf{i}_1 \rho_2 + \mathbf{i}_2 \left[(1 + \lambda h_1) \sin\varphi - \rho \frac{K}{H_1 \sin\varphi} \right] \right\} \omega^2, \tag{13}$$

where

$$d\rho = \rho_1 \omega^1 + \rho_2 \omega^2, \qquad \Delta l^\alpha_{11} = l^\alpha_{111} \omega^1, \qquad \Delta l^\alpha_{22} = l^\alpha_{222} \omega^2. \tag{14}$$

Hence condition (12) is satisfied in two ways:

$$1)\ \omega^2 = 0, \quad \rho = 0; \qquad 2)\ \omega^1 = 0, \quad \rho = \frac{(1 + \lambda h_1) H_1 \sin^2\varphi}{K}. \tag{15}$$

The first solution gives the focus M_1 and the focal family $\omega^2 = 0$, the second—the focus

$$M_2 = M_1 + \mathbf{i}_1 \frac{(1 + \lambda h_1) H_1 \sin^2\varphi}{K} \tag{16}$$

and the family $\omega^1 = 0$.

5. The net $\omega^2 = 0$, $\omega^1 = 0$ on the surface (M_1), corresponding to the Voss net, is orthogonal and conjugate. However, the surface (M_1) will not be a Guichard surface, since the focal net on the surface (M_2) is not orthogonal. Indeed, substituting ρ from formula (15) into (13), we obtain

$$dM_2 = \left\{ \mathbf{i}_1(\lambda H_1 + \rho_1) + \frac{1 + \lambda h_1}{K} \sin\varphi \left(H_1^2 \frac{\mathbf{e}_1 - \mathbf{e}_2 \cos\varphi}{\sin\varphi} + \frac{l^4_{11} l^{[3]}_{11}}{H_1} \frac{l^3_{11}\mathbf{e}_4 - l^4_{11}\mathbf{e}_3}{H_1} \right) \right\} \omega^1 + \mathbf{i}_1 \rho_2 \omega^2,$$

and the condition of orthogonality of the focal net $(\lambda H_1 + \rho_1)\rho_2 = 0$, for $\rho_2 \neq 0$, is equivalent to

$$[\lambda H_1 \omega^1 + d\rho, \omega^2] = 0,$$

or, after substituting $d\rho$ from (15):

$$K\,(l^3_{11} l^3_{111} + l^4_{11} l^4_{111}) - H_1^2\,(l^3_{22} l^3_{111} + l^4_{22} l^4_{111}) = 0$$

or

$$(l^3_{11}l^4_{22} - l^4_{11}l^3_{22})\,(l^4_{11}l^3_{111} - l^3_{11}l^4_{111}) = 0.$$

The vanishing of the first factor leads to the coincidence of the asymptotic forms Φ^3, Φ^4 and to a three-dimensional enveloping space. The vanishing of the second factor is written in the equivalent form

$$l^4_{11}\Delta l^3_{11} - l^3_{11}\Delta l^4_{11} = 0$$

or

$$\omega^3_4 = d\,\mathrm{arctg}\,\frac{l^3_{11}}{l^4_{11}},$$

whence, by exterior differentiation, we obtain

$$[\omega^1_4\omega^3_1] + [\omega^2_4\omega^3_2] = 0,$$

and again

$$l^3_{11}l^4_{22} - l^4_{11}l^3_{22} = 0.$$

Thus, the connection established by Guichard between Voss surfaces and Guichard surfaces exists only in three-dimensional space.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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