MATHEMATICS

V. P. KHAVIN

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. P. KHAVIN

ANALYTIC CONTINUATION OF POWER SERIES AND FABER POLYNOMIALS

(Presented by Academician V. I. Smirnov on 1 IX 1957)

Let G be a simply connected domain of the complex plane, containing the point $z = \infty$ in its interior; let K be a closed bounded set complementary to G; assume that K contains more than one point. Let $\Phi(z)$ be the function giving a conformal mapping of G onto the exterior of the circle $|w| = \rho$ in such a way that, in a neighborhood of the infinitely distant point,

$$\Phi(z) = z + \alpha_0 + \frac{\alpha_1}{z} + \cdots.$$

For an integer nonnegative k,

$$[\Phi(z)]^k = \Phi_k(z) + \frac{\alpha_1^{(k)}}{z} + \frac{\alpha_2^{(k)}}{z^2} + \cdots,$$

where $\Phi_k(z)$ is the so-called Faber polynomial of degree k ([1]).

Let

$$\varphi(z) = \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_n}{z^n} + \cdots \tag{1}$$

be a series convergent in some neighborhood of the infinitely distant point.

The aim of the present note is to prove the following theorem:

Theorem 1. Let

$$\Phi_k(z) = c_0^{(k)} + c_1^{(k)} z + \cdots + c_k^{(k)} z^k.$$

In order that the series (1) converge to a function analytic in G, it is necessary and sufficient that the condition

$$\overline{\lim}_k \left| c_0^{(k)} b_1 + c_1^{(k)} b_2 + \cdots + c_k^{(k)} b_{k+1} \right|^{1/k} \leqslant \rho. \tag{2}$$

be satisfied.

Let B be the totality of all functions analytic on K. We shall say that a sequence $\{\varphi_n\}_1^\infty \subset B$ converges to zero if there exists a number $r > \rho$ such that $\varphi_n(z)$, $n = 1, 2, ...$, are analytic up to the curve L_r, which is the preimage of the circle $|w| = r$ under the mapping $w = \Phi(z)$, and $\lim_{n\cdot\infty} \varphi_n(z) = 0$ uniformly on L_r.

The convergence introduced in B cannot be metrized. This convergence introduces in B the so-called locally convex topology of the inductive limit of normed (complete) spaces ([3]).

The proof of Theorem 1 is based on the following two lemmas.

Lemma 1 (Köthe ([2])). *The general form of a continuous linear functional in B is given by the formula*

$$F(\psi) = \frac{1}{2\pi i} \int_{L_{r(\psi)}} \varphi_F(z)\psi(z)\, dz, \qquad \psi \in B. \tag{3}$$

Here $\varphi_F(z)$ is a function analytic in G, equal to zero for $z = \infty$,

completely determined by the functional; $r(\psi) > \rho$ is a number such that $\psi(z)$ is analytic up to the curve $L_r(\psi)$.

Lemma 2. Let the sequence of complex numbers $\{q_n\}_{n=0}^\infty$ be such that the series

$$\sum_{s=0}^{\infty} q_s \lambda_s \tag{4}$$

converges, whatever the sequence $\{\lambda_n\}_{n=0}^\infty$ with

$$\overline{\lim} \sqrt[n]{|\lambda_n|} < \frac{1}{\rho};$$

then

$$\overline{\lim} \sqrt[n]{|q_n|} \le \rho. \tag{5}$$

Conversely, if (5) is satisfied, then the series (4) converges for every sequence $\{\lambda_n\}_{n=0}^\infty$,

$$\overline{\lim} \sqrt[n]{|\lambda_n|} < \frac{1}{\rho}.$$

We now turn to the proof of Theorem 1.

Sufficiency. If $\psi \in B$, then

$$\psi(z)=\sum_{m=0}^{\infty}\lambda_m\Phi_m(z),\qquad \overline{\lim}\sqrt[m]{|\lambda_m|}<\frac{1}{\rho},$$

and the series converges uniformly on $L_r(\psi)$; such an expansion is unique ([1]).

The series (1), converging in a neighborhood of the point at infinity, makes it possible to define, on the set of polynomials dense in B, an additive and homogeneous functional:

$$F^*(p)=\frac{1}{2\pi i}\int_{|z|=R}\varphi(z)p(z)\,dz$$

($p(z)$ is an arbitrary polynomial; $R>0$ is such that the series (1) converges for $|z|=R/2$).

Let

$$p_N(z)=\sum_{m=0}^{N}\lambda_m\Phi_m(z).$$

Then

$$F^*(p_N)=\sum_{m=0}^{N}\lambda_m\frac{1}{2\pi i}\int_{|z|=R}\Phi_m(z)\sum_{s=1}^{\infty}\frac{b_s}{z^s}\,dz=$$

$$=\sum_{m=0}^{N}\lambda_m\left(c_0^{(m)}b_1+c_1^{(m)}b_2+\ldots+c_m^{(m)}b_{m+1}\right).$$

If (2) is satisfied, then, by Lemma 2, we conclude that the series

$$\sum_{m=0}^{\infty}\lambda_m\left(c_0^{(m)}b_1+c_1^{(m)}b_2+\ldots+c_m^{(m)}b_{m+1}\right)$$

converges if

$$\overline{\lim}\sqrt[m]{|\lambda_m|}<\frac{1}{\rho}.$$

Then the functional $F^*(\psi)$ extends to B with preservation of additivity and homogeneity. Namely, if $\psi\in B$,

$$\psi(z)=\sum_{m=0}^{N}\lambda_m(\psi)\Phi_m(z),\qquad \overline{\lim}\sqrt[m]{|\lambda_m|}<\frac{1}{\rho},$$

then set

$$F(\psi) = \lim_{N\to\infty} F^*(p_N) = \lim_{N\to\infty} \sum_{m=0}^{N} \lambda_m(\psi)(\Phi, b)_m \tag{6}$$

$$\left((\Phi, b)_m = \sum_{s=0}^{m} c_s^{(m)} b_{s+1} \right).$$

The functional F, which realizes the extension F^* to B, is continuous in B. Indeed, the coefficients $\lambda_m = \lambda_m(\psi)$, $m = 0, 1, 2, \ldots$, are continuous functionals in B. We shall use the Banach-Steinhaus theorem, which asserts that a linear functional defined on the inductive limit of complete normed spaces and which is the weak limit of continuous linear functionals is continuous (3).

From this fact and from (6) the continuity of F follows immediately. Then (3) holds. If $p(z)$ is a polynomial, then $F^*(p) = F(p)$,

$$\int_{|z|=R} (\varphi_F(z) - \varphi(z))p(z)\, dz = 0,$$

and therefore $\varphi_F(z) = \varphi(z)$ for $|z| > R$. The function $\varphi_F(z)$ gives an analytic continuation of $\varphi(z)$ in G.

Necessity. If $\varphi(z)$ is analytically continuable in G, then the functional (3) is continuous in B ($\varphi_F = \varphi$). If

$$\overline{\lim} \sqrt[m]{|\lambda_m|} < \frac{1}{\rho},$$

then $\psi(z) = \sum_{m=0}^{\infty} \lambda_m \Phi_m(z)$ is an element of B, and the series converges in the sense of convergence in B (1). Then

$$F(\psi) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{L_r(\psi)} \varphi(z) p_N(z)\, dz = \lim_{N\to\infty} \sum_{m=0}^{N} \lambda_m (\Phi, b)_m.$$

By Lemma 2,

$$\overline{\lim} \sqrt[m]{|(\Phi, b)_m|} \leqslant \rho.$$

The theorem is proved.

Theorem 2. The following assertions are equivalent:

1) The series $\sum_{s=1}^{\infty} b_s z^s$ converges for $|z| < 1$ to an analytic function having no nonreal singularities in the rectilinear star with center at the origin.

2)

$$\overline{\lim}\left|c_0^{(m)}b_1 + c_1^{(m)}b_2 + \ldots + c_m^{(m)}b_{m+1}\right|^{1/m} \leqslant \frac{1}{2},$$

where

$$\sum_{s=0}^{m} c_s^{(m)} z^s = 2^{1-m}\cos m \arccos z.$$

This theorem follows immediately from the preceding one, if one takes into account that in the case when $K = [-1, +1]$,

$$\Phi_m(z) = \frac{1}{2^{m-1}}\cos m \arccos z$$

(the Chebyshev polynomial).

In conclusion I offer my sincere gratitude to Academician V. I. Smirnov and Professor S. M. Lozinskii for their attention to this note.

Leningrad State University
named after A. A. Zhdanov

Received
15 VII 1957

References

1. A. I. Markushevich, *Theory of Analytic Functions*, 1950.

2. G. Köthe, *Math. Zs.*, **57**, 13 (1952).

3. N. Bourbaki, *Éléments de mathématique*, Livre 5, Paris, 1955.

Note: Figure translations are in progress. See original paper for figures.