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# MATHEMATICS

1958

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**Abstract**

**Full Text**

**MATHEMATICS**

**V. M. VOLOSOV**

**ASYMPTOTICS OF INTEGRALS OF CERTAIN PERTURBED SYSTEMS**

*(Presented by Academician N. N. Bogolyubov, 22 IV 1958)*

§ 1. **Statement of the problem.** Consider the system of equations

$$\dot{x}_0 = M(x_0, y_0, \vec{\mu}_0), \quad \dot{y}_0 = N(x_0, y_0, \vec{\mu}_0), \quad (1)$$

where  $x_0, y_0$  are unknown functions;  $\vec{\mu}_0 = \{\mu_{10}, \mu_{20}, \dots, \mu_{n0}\}$  is a set of  $n$  independent parameters. We shall call (1) the **unperturbed system**. Suppose that in some domain the general solution of (1) is a family of periodic solutions depending on arbitrary constants and on  $\vec{\mu}_0$ . Let  $\Phi(x_0, y_0, \vec{\mu}_0)$  be a certain continuous, bounded, sufficiently smooth integral of system (1), not containing  $t$ . Along a cycle of the unperturbed system  $\Phi = c_0 = \text{const}$ . With respect to the functions  $M, N$  we assume that they are continuous, bounded, and sufficiently smooth in the domain considered below. We shall assume that in this domain the conditions for existence and uniqueness of solutions of (1) and for continuous dependence of the solutions on the initial conditions and on  $\vec{\mu}_0$  are satisfied. Suppose that there exist an open domain  $\{\mu\}$  of the space  $\mu_1, \mu_2, \dots, \mu_n$  and an open domain  $G$  of the phase plane  $(x, y)$  such that the phase trajectory  $x_0(t, \vec{\mu}_0), y_0(t, \vec{\mu}_0)$  of system (1), starting at any point of the closure  $\bar{G}$  of the domain  $G$ , for values of  $\vec{\mu}_0$  from the closure  $\bar{\{\mu\}}$  of the domain  $\{\mu\}$ , is a closed curve and corresponds to some periodic solution of (1). Assume that the family of periodic solutions of (1), whose phase trajectories start in  $\bar{G}$ , has the integral  $\Phi = c_0$ , and moreover assume that this family can be written in the form

$$x_0 = x_0(c_0, \vec{\mu}_0, \omega(c_0, \vec{\mu}_0)t + h), \quad y_0 = y_0(c_0, \vec{\mu}_0, \omega(c_0, \vec{\mu}_0)t + h), \quad (2)$$

where  $c_0$  is an arbitrary constant corresponding to the integral  $\Phi(x_0, y_0, \vec{\mu}_0) = c_0$ ;  $h$  is likewise an arbitrary constant;  $\omega = 2\pi/T$  is the frequency of oscillations;  $T(c_0, \vec{\mu}_0)$  is the period of the solutions. It is assumed that  $x_0, y_0$  are periodic functions of the argument  $\omega t + h$  with period  $2\pi$ , and that, for the values  $c_0, \vec{\mu}_0$  considered below, the function  $\omega$  is bounded, continuous, sufficiently smooth, and  $\omega \geq \alpha > 0$  ( $\alpha = \text{const}$ ). Consider the system of equations

$$\dot{x} = M(x, y, \bar{\mu}) + \varepsilon f^{(y)}(x, y, \bar{\mu}), \quad \dot{y} = N(x, y, \bar{\mu}) - \varepsilon f^{(x)}(x, y, \bar{\mu}),$$

$$\dot{\bar{\mu}} = \varepsilon \bar{\varphi}(x, y, \bar{\mu}), \quad (3)$$

where  $\varepsilon$  is a small parameter; this system will be called the **perturbed system**. The functions  $\varepsilon f^{(x)}$  and  $\varepsilon f^{(y)}$  are small perturbations. The last equation of the system

- (3) describes a slow change of the parameter  $\bar{\mu}$  with a small velocity  $\varepsilon \bar{\varphi}$ , depending on the state of system (3) (in the unperturbed system (1),  $\bar{\mu} = \bar{\mu}_0 = \text{const}$ );  $x, y$  in system (3) are the coordinates of the perturbed motion. Concerning the functions  $f^{(x)}, f^{(y)}, \bar{\varphi}$ , we assume that they are continuous, bounded, and sufficiently smooth in the domain  $\bar{\Gamma} = \bar{G} + \{\bar{\mu}\}$  of the variables  $x, y, \mu_1, \mu_2, \dots, \mu_n$ .

Let  $G_0$  be some open subdomain of the domain  $G$ , lying entirely inside  $G$  together with its boundary; let  $\{\bar{\mu}\}_0$  be an open subdomain of the domain  $\{\bar{\mu}\}$ , lying together with its boundary inside  $\{\bar{\mu}\}$ . Construct the domain  $\Gamma_0 = G_0 + \{\bar{\mu}\}_0$  of the variables  $x, y, \mu_1, \mu_2, \dots, \mu_n$ . We shall consider integral curves of system (3) beginning at interior points of the domain  $\Gamma_0$ . Let  $\varepsilon > 0$  be arbitrarily small, but fixed. Introduce the time interval  $[t_1, t_1 + a/\varepsilon]$ , where  $a > 0$  is an arbitrarily large but fixed number;  $t_1$  is an arbitrary instant of time. We now take integral curves of (3) beginning at some instant  $t = t_0 \in [t_1, t_1 + a/\varepsilon]$  inside  $\Gamma_0$ , and shall consider them either on the whole interval  $[t_1, t_1 + a/\varepsilon]$ , if for  $t_1 \leq t \leq t_1 + a/\varepsilon$  the corresponding phase trajectory does not leave  $\Gamma_0$ , or on some part  $[t_0 - b, t_0 + c] \subset [t_1, t_1 + a/\varepsilon]$  ( $b, c > 0$ ) such that for  $t \in [t_0 - b, t_0 + c]$  the trajectory remains inside  $\Gamma_0$ .

If in the integral  $\Phi(x_0, y_0, \bar{\mu}_0) = c_0$  one substitutes, in place of  $x_0, y_0, \bar{\mu}_0$  ( $\bar{\mu}_0 = \text{const}$ ), the solutions  $x, y, \bar{\mu}$  of system (3) that we are considering, then  $\Phi$  will, in general, become a variable quantity:  $\Phi(x, y, \bar{\mu}) = c(t, \varepsilon)$ . The problem consists in investigating the asymptotics of the function  $c(t, \varepsilon) = \Phi(x, y, \bar{\mu})$  on the time interval mentioned. It is assumed that in the domain  $\bar{\Gamma}$  the perturbed system (3) satisfies the conditions for existence and uniqueness of solutions and for continuous dependence of the solutions on the initial values. For the integral  $c = \Phi$  and the slowly varying parameters  $\bar{\mu} = \{\mu_1, \mu_2, \dots, \mu_n\}$ , we shall derive zero-approximation equations describing these quantities with an error  $\sim \varepsilon$  on the entire indicated time interval (generally speaking, according to what was said above, an interval in  $t$  of size  $\sim 1/\varepsilon$  is considered).

**§ 2. Main results.** Suppose that in the domain  $\bar{G}$  there exists a continuous, bounded, and sufficiently smooth integrating multiplier  $\lambda(x_0, y_0, \bar{\mu}_0)$  of system (1), corresponding to the integral  $\Phi = c_0$ ; this means that  $\partial\Phi/\partial y = M\lambda$ ,  $\partial\Phi/\partial x = -N\lambda$ . In system (3) make the change of variables according to the formulas

$$x = x_0(c, \bar{\mu}, \psi), \quad y = y_0(c, \bar{\mu}, \psi), \quad \dot{\bar{\mu}} = \bar{\mu}, \quad (4)$$

where  $c, \bar{\mu}, \psi$  are unknown functions;  $x_0, y_0$  are periodic functions of  $\psi$  of period  $2\pi$ , taken from (2). After a number of transformations, for  $c, \bar{\mu}, \psi$  one obtains a system of equations, from which we write here only two:

$$\dot{c} = \varepsilon \{ (f^{(x)} x'_\psi + f^{(y)} y'_\psi) \lambda + \bar{\varphi}(\bar{\nabla}_\mu \Phi) \}, \quad \dot{\bar{\mu}} = \varepsilon \bar{\varphi}, \quad (5)$$

where the right-hand sides are expressed in terms of  $c, \bar{\mu}, \psi$  by means of (4);  $\bar{\nabla}_\mu = \left\{ \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \mu_2}, \dots, \frac{\partial}{\partial \mu_n} \right\}$ . System (5) belongs to the type of systems with rapidly rotating-

ing phase (together with the equation for  $\psi$ , which is not written), studied by N. N. Bogolyubov and D. N. Zubarev <sup>(2)</sup>. To obtain the equations of the zeroth approximation according to <sup>(2)</sup>, it is necessary to average the right-hand sides of (5) with respect to  $\psi$ . As a result of averaging, after some transformations, the equations of the zeroth approximations for  $c, \bar{\mu}$  are brought to the form

$$\dot{c} = \frac{\varepsilon}{T} \oint \lambda (f^{(x)} dx_0 + f^{(y)} dy_0) + \frac{\varepsilon}{T} \int_T \varphi(\bar{\nabla}_\mu \Phi) dt, \quad \dot{\bar{\mu}} = \frac{\varepsilon}{T} \int_T \varphi dt \equiv \varepsilon \bar{\varphi} \quad (6)$$

(for the zeroth approximations  $c, \bar{\mu}$  we have not introduced new notation).

Equations (6) constitute the main result of the present work. These equations describe the change of  $c, \bar{\mu}$  on the interval  $t \sim 1/\varepsilon$  with an error  $\sim \varepsilon$ . The quadratures on the right-hand sides of (6) are taken along the cycles (2) of the unperturbed system (1), where in place of  $c = \text{const}$  and  $\bar{\mu} = \text{const}$  one substitutes the desired zeroth approximations for  $c, \bar{\mu}$ . The solutions of (6) vary slowly (since  $\dot{c}, \dot{\bar{\mu}} \sim \varepsilon$ ), and, since the solution (2) and the integral  $\Phi$  of system (1) are assumed known, the integration of (6) is in principle simpler than the integration of (3). The second equation (6) means that, in the zeroth approximation, the true velocity  $\varepsilon \varphi$  of change of the parameter  $\bar{\mu}$  can be replaced by the mean over a period value of the velocity  $\varepsilon \bar{\varphi}$ .

The averaging of systems with a rapidly rotating phase is considered in <sup>(2)</sup> as a formal transformation of the system. However, with the aid of N. N. Bogolyubov's methods <sup>(1)</sup> it is not difficult to formulate conditions under which this averaging indeed leads to the equations of the zeroth approximation. Without formulating these conditions here, let us assume that the systems (5), (6) satisfy the conditions of existence and uniqueness of solutions and of continuous dependence of solutions on initial values, and that such averaging is applicable. One can also find higher approximations for  $c, \bar{\mu}$  and approximations for  $\psi$ —the phase of the oscillations. Then, with the aid of (4), one can obtain the asymptotics of

the solutions  $x, y, \bar{\mu}$  of system (3). The equations of higher approximations are not given here, since in the general case they are very cumbersome.

§ 3. **Some special cases (canonical systems).** Consider the canonical system

$$\dot{q}_0 = \frac{\partial H}{\partial p}(q_0, p_0, \bar{\mu}_0), \quad \dot{p}_0 = -\frac{\partial H}{\partial q}(q_0, p_0, \bar{\mu}_0), \quad (7)$$

which is a special case of (1). The corresponding perturbed system has the form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p, \bar{\mu}) - \varepsilon f^p(q, p, \bar{\mu}), & \dot{p} &= -\frac{\partial H}{\partial q}(q, p, \bar{\mu}) + \varepsilon f^q(q, p, \bar{\mu}), \\ \dot{\bar{\mu}} &= \varepsilon \varphi(q, p, \bar{\mu}). \end{aligned} \quad (8)$$

System (7) has the integral of total energy

$$E = H(q_0, p_0, \bar{\mu}_0) = \text{const.}$$

Apply formula (6) to (8), putting  $\lambda \equiv 1$ , i.e., taking  $E$  as the integral under study. This leads to the equations

$$\dot{E} = \frac{\varepsilon}{T} \oint (f^q dq_0 + f^p dp_0) + \frac{\varepsilon}{T} \int_T \varphi(\bar{\nabla}_\mu H) dt, \quad \dot{\bar{\mu}} = \varepsilon \bar{\varphi}. \quad (9)$$

(9) describes the change of energy: the rate of its change is equal to the mean power of the perturbing forces  $\varepsilon f^q, \varepsilon f^p$  over a period, added to the mean power expended on changing the parameter  $\bar{\mu}$ . (9) is a special case of (6), since the general system (3) corresponds to the unperturbed system (1),

which may or may not be canonical. Equation (9) was given in my paper <sup>9</sup>. Later, independently of <sup>9</sup>, the energy equation for (8) was derived by G. S. Makaeva. The general equation (6) has not previously been published.

For systems (8), the asymptotics of the action integral

$$I = \oint p_0 dq_0,$$

taken over a cycle of the unperturbed motion (7), is of interest. In this case equation (6) reduces to the following special form:

$$\dot{I} = \varepsilon \oint (f^q dq_0 + f^p dp_0) + \varepsilon \int_T (\bar{\varphi} - \bar{\bar{\varphi}}) \bar{\nabla}_\mu H dt, \quad \dot{\bar{\mu}} = \varepsilon \bar{\varphi}. \quad (10)$$

(10) describes the change of the action integral: the rate of its change is equal to the work of the perturbing forces over a period, combined with the virtual work expended in changing  $\vec{\mu}$  with velocity  $\varepsilon(\vec{\varphi} - \bar{\vec{\varphi}})$  ( $\varepsilon(\vec{\varphi} - \bar{\vec{\varphi}})$  is the deviation of the velocity  $\vec{\mu}$  from its mean value\*). If the parameters  $\vec{\mu}$  change uniformly, i.e.  $\dot{\vec{\mu}} = \varepsilon\vec{\varphi} = \text{const}$ , and the work  $\oint(f^q dq_0 + f^p dp_0) = 0$  on every cycle (7), then, according to (10),  $\dot{I} = 0$  and  $I = \text{const}$ . Integrals  $c = \Phi$  for which, in the zero approximation,  $\dot{c} = 0$ , are called **adiabatic invariants**. Under the indicated conditions, the action integral is an adiabatic invariant. In the physical literature (for example <sup>6,7</sup>) the invariance of  $I$  is known when (8) has canonical form. The condition  $\oint(f^q dq_0 + f^p dp_0) = 0$  on every cycle (7) is a weaker restriction, since in order that system (8) could be regarded as canonical it is necessary that  $\oint(f^q dq + f^p dp) = 0$  on every closed contour. It follows from (6) that for systems (8) of general type, or for (3), the integral  $I$  may or may not be an invariant. A general system (3) in some cases may have an adiabatically invariant integral  $c = \Phi$ , if the right-hand side of (6) vanishes identically.

The integrals  $\Phi_1(q_0, p_0, \vec{\mu}_0)$ ,  $\Phi_2(q_0, p_0, \vec{\mu}_0)$  of system (7), not containing  $t$ , are functionally dependent:  $\Psi(\Phi_1, \Phi_2, \vec{\mu}_0) = 0$ . If  $\vec{\nabla}_{\vec{\mu}} \Psi = 0$ , then we shall identify  $\Phi_1, \Phi_2$ ; otherwise we shall regard them as different integrals. It can be shown that it follows from (6) that for systems (8), under the conditions  $\dot{\vec{\mu}} = \varepsilon\vec{\varphi} = \text{const}$  and  $\oint(f^q dq_0 + f^p dp_0) = 0$ , the action integral  $I$  is, in the indicated sense, the unique adiabatic invariant (the whole class of systems (8) with arbitrary  $H, f^q, f^p$  is meant; for individual systems this rule may be violated).

Various problems connected with the asymptotics of solutions of systems similar in type to (3) were considered in <sup>3-5</sup>.

The author takes this opportunity to express gratitude to Acad. N. N. Bogolyubov for his attention to the present work.

Moscow State University  
named after M. V. Lomonosov

Received  
18 IV 1958

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\* Part of the results of my papers <sup>8,9</sup> follows from (10).

*Note: Figure translations are in progress. See original paper for figures.*

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