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Abstract

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ON THE PRINCIPLE OF REDUCTION

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In the present article a method is given for investigating the problem of stability of the zero solution of a system of $n+k$ ordinary differential equations. This method consists in studying systems of k-th and n-th orders obtained from the original equations. The application of this method makes it possible to advance the solution in a number of doubtful cases.

Consider the system of ordinary differential equations

$$\begin{aligned} \frac{dy_s}{dt} &= f_s(t, x_1, \dots, x_n, y_1, \dots, y_k) \qquad (s = 1, \dots, k), \\ \frac{dx_j}{dt} &= g_j(t, x_1, \dots, x_n, y_1, \dots, y_k) \qquad (j = 1, \dots, n). \end{aligned} \tag{1}$$

We shall assume that the right-hand sides of system (1) are given in the domain $t \geqslant 0$, $|X| < H$, $|Y| < H$, and are continuous there. Here $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_k)$, $|X| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $H > 0$. Suppose further that $f_s \equiv 0$ when $Y = 0$, and $g_j \equiv 0$ when $X = Y = 0$.

Definition 1. The zero solution of system (1) will be called **stable** if for every $\varepsilon > 0$ one can indicate a $\delta(\varepsilon) > 0$ such that, for $|X^{(0)}| < \delta$, $|Y^{(0)}| < \delta$, one has

$$|X(t; X^{(0)}, Y^{(0)}, t_0)| < \varepsilon, \qquad |Y(t; X^{(0)}, Y^{(0)}, t_0)| < \varepsilon$$

for $0 \leqslant t_0 \leqslant t$. If, in addition, we have $X(t, X^{(0)}, Y^{(0)}, t_0) \to 0$, $Y(t, X^{(0)}, Y^{(0)}, t_0) \to 0$ as $t \to +\infty$, then the zero solution of system (1) will be called **asymptotically stable**. Here $X(t, X^{(0)}, Y^{(0)}, t_0)$, $Y(t, X^{(0)}, Y^{(0)}, t_0)$ denote the solution of system (1) with initial data $X = X^{(0)}$, $Y = Y^{(0)}$ at $t = t_0$.

If in the first group of equations entering into system (1) the quantities $x_1, \dots, x_n$ are replaced by arbitrary continuously differentiable functions $x_1(t), \dots, x_n(t)$, defined for $t \geqslant 0$ and satisfying the condition $|X(t)| < H$, then we obtain a system of k equations

$$\frac{dy_s}{dt} = f_s(t, x_1(t), \dots, x_n(t), y_1, \dots, y_k) \qquad (s = 1, \dots, k), \tag{2}$$

for which the point $Y = 0$ will be an equilibrium position.

Definition 2. The zero solution of system (2) will be called **strongly stable** if one can indicate a number $H_1 > 0$ such that for every $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ with the following property: if $|Y^{(0)}| \leqslant \delta_1$

will have $|Y(t, Y^{(0)}, t_0)| < \varepsilon_1$ for $0 \leq t_0 \leq t$ for every function $X(t)$ such that $|X(t)| \leq H_1$. If, in addition, $Y(t, Y^{(0)}, t_0) \to 0$ as $t \to +\infty$, the zero solution of system (2) shall be called **strongly asymptotically stable**.

Consider a function $W(t, X, Y)$ such that $W \equiv 0$ when $X = Y = 0$.

Definition 3. The function $W(t, X, Y)$ will be called **strictly negative definite with respect to** X if there exists a continuous function $f(r)$ such that: 1) $f = 0$ for $r = 0$ and $f > 0$ for $r > 0$; 2) for every vector-function $Y(X)$ satisfying the inequality $|Y(X)| \leq f(|X|)$, the function $W(t, X, Y(X))$ is negative definite with respect to X.

Theorem 1. *If: 1) the zero solution of system (2) is strongly stable (strongly asymptotically stable); 2) in the domain $|X| < H_2$, $t \geq 0$ there exists a continuously differentiable positive-definite function $V(t, X)$, $V(t, X) \to 0$ as $X \to 0$ uniformly in $t \geq 0$; 3) the function*

$$W(t, X, Y) = \frac{\partial V}{\partial t} + \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} g_j(t, X, Y)$$

is strictly negative definite with respect to X, then the zero solution of system (1) will also be stable (asymptotically stable).

Theorem 2. *If: 1) the functions $f_s(t, X, Y)$ can be expanded in convergent series in integral positive powers of the quantities $x_1, \dots, x_n, y_1, \dots, y_k$ with real, continuous, bounded coefficients given for $t \geq 0$, and the zero solution of system (2) is stable (asymptotically stable) independently of the choice of terms of order n (1); 2) the functions $g_j(t, X, 0)$ are continuously differentiable in all arguments for $t \geq 0$ and $|X| < H$, and the zero solution of the system*

$$\frac{dx_j}{dt} = g_j(t, X, 0) \tag{3}$$

is uniformly asymptotically stable and uniformly attracting [2,3]; *3) the partial derivatives of the functions $g_j(t, X, 0)$ are bounded together with the functions $g_j(t, X, Y) - g_j(t, X, 0)$ for $t \geq 0$, $|X| \leq H$, $|Y| \leq H$, then the zero solution of system (1) is stable (asymptotically stable).*

Corollary. *If: 1) condition 1 of Theorem 2 is satisfied; 2) the functions $g_j(t, X, 0) = g_j(X, 0)$ do not depend explicitly on t, and the zero solution of system (3) is asymptotically stable; 3) the functions $g_j(X, 0)$ are continuously differentiable, and the functions $g_j(t, X, Y) - g_j(X, 0)$ are bounded for*

$t \geq 0$, $|X| \leq H$, $|Y| \leq H$, then the zero solution of system (1) is stable (asymptotically stable).

The theorems stated above in some cases make it possible to simplify the solution of the question of stability of the zero solution of system (1), if even the expansions of the functions $g_j(t, X, 0)$ contain no terms linear with respect to $x_1, \dots, x_n$.

As an example, consider the system

$$\frac{dy_s}{dt} = \sum_{m=1}^{+\infty} Y_s^{(m)}(t, X, Y) = f_s(t, X, Y), \tag{4}$$

$$\frac{dx_j}{dt} = \sum_{m=0}^{+\infty} X_j^{(m)}(t, X, Y) = g_j(t, X, Y), \tag{5}$$

where $Y_s^{(m)}$, respectively $X_j^{(m)}$, are homogeneous forms of degree m with respect to the quantities $y_1, \dots, y_k$, respectively $x_1, \dots, x_n$.

We shall assume that the coefficients of these forms are convergent power series with respect to the quantities $x_1, \dots, x_n$, respectively $y_1, \dots, y_k$,

whose coefficients are real, continuous, and bounded for $t \geq 0$. Let the system (3)–(4) have the zero solution. Suppose that

$$g_j(t, X, 0) = \sum_{m=\mu}^{+\infty} \overline{X}_j^{(m)}(t, X)$$

and the functions $\overline{X}_j^{(\mu)}$ are homogeneous forms of degree μ with real constant coefficients.

Let us now consider the system

$$\frac{dx_j}{dt} = \overline{X}_j^{(\mu)}(X). \tag{6}$$

Theorem 3. *If: 1) the zero solution of system (4) is stable (asymptotically stable) independently of the choice of terms of order higher than n ([1]); 2) the zero solution of system (6) is asymptotically stable, then the zero solution of system (4)–(5) is stable (asymptotically stable).*

Let us examine condition 2) of Theorem 3 in more detail. In order that the zero solution of system (6) be asymptotically stable, it is necessary and sufficient ([6]) that the system of equations

$$\sum_{j=1}^{n} \frac{\partial Y}{\partial x_j} \overline{X}_j^{(\mu)} = W,$$

$$\sum_{j=1}^{n} \frac{\partial V}{\partial x_j} x_j = (m+1-\mu)V, \tag{7}$$

where W is a positive definite function, homogeneous of order $m > \mu - 1$, have a solution in the form of a negative definite function $V(X)$. We note that for $n = 2$ the system (7) can easily be solved in finite form, i.e. the function $V(X)$ is expressed in terms of $\overline{X}_1^{(\mu)}$, $\overline{X}_2^{(\mu)}$, and W by means of a finite number of quadratures. For $n > 2$ the solution of system (7) can also be found in finite form, but in this case it has a rather complicated form. For $n = 2$, condition 2) of Theorem 3 can also be verified with the aid of Kamenkov' s theorem ([4]).

We note that such a method for solving the question of stability was first applied by A. M. Lyapunov ([5]) in the investigation of doubtful cases.

In ([1]) a development of this method was given for solving the question of stability in the case of analytic right-hand sides of system (1) of a special form. In the present work the method of proof contained in ([1]) has been essentially used.

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Note: Figure translations are in progress. See original paper for figures.