A. V. MALYSHEV

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Abstract

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MATHEMATICS

A. V. MALYSHEV

ON THE REPRESENTATION OF LARGE NUMBERS BY POSITIVE TERNARY QUADRATIC FORMS OF ODD RELATIVELY PRIME INVARIANTS

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Let $f(x_1, x_2, \ldots, x_n)$ be an integral quadratic form of determinant d. There is a known theorem (^{1}) stating that, if an integer m satisfies certain necessary conditions of the type

$$f(x_1, x_2, \ldots, x_n) \equiv m \pmod{8d},$$

then this number is representable by at least one of the forms of the genus of f; there, too, a formula is given for the number of such representations. Much more complicated is the question of the representation of the number m by each form f. In this case one should expect more or less simple results only for forms in three or more variables, since for representability by binary forms the necessary conditions of the type indicated above prove to be clearly insufficient. For indefinite forms the problem is, to a considerable extent, exhausted by the theorems of Meyer (^{2},^{3}), who proved that under very broad assumptions a genus of indefinite quadratic forms in three and more variables contains only one class (see also (^{4},^{5})). For positive quadratic forms in four and more variables V. A. Tartakovskii (^{6}), by analytic methods, constructed an asymptotic formula for the number of representations of the number m by the form f, so that, when the necessary conditions of the type indicated above are fulfilled, all sufficiently large numbers m are representable by the form f. Yu. V. Linnik (^{7}), with the aid of the arithmetic of quaternions (and its generalizations), proved a number of theorems on the representation of numbers by certain ternary quadratic forms. In developing these investigations we prove the following theorems.

Theorem 1. Let $f(x, y, z)$ be an integral primitive positive ternary quadratic form of odd relatively prime invariants $[\Omega, \Delta]$ (Ω is the greatest common divisor of the coefficients of the form $\bar{f}$, reciprocal to f; $\Omega^2\Delta = \det f$); let m be an integer relatively prime to $2\Omega\Delta$, for which the congruence

$$f(x, y, z) \equiv m \pmod{8\Omega\Delta} \tag{1}$$

is solvable in integers x, y, z. Then there exists a number s_0, depending only on Ω, Δ, such that if m is divisible by the square of an integer $s \leq s_0$, then m is primitively representable by the form f. Moreover, if by $t(f,m)$ we denote the number of primitive representations of the number m by the form f, then

$$t(f,m) > \varkappa h(-\Delta m), \tag{2}$$

where $h(-\Delta m)$ is the number of classes of positive binary properly primitive quadratic forms of determinant Δm; $\varkappa > 0$ is a constant depending only on Ω, Δ.

This theorem is a refinement of a result of Yu. V. Linnik (^{8}), where, under the same conditions, the existence of a representation of the number m by the form f is proved, but not necessarily a primitive one (and no estimates are given). Apart from its independent interest, it is used essentially in the proof of Theorem 2.

Theorem 2. Let $f(x,y,z)$ be an integral primitive positive ternary quadratic form of odd relatively prime

invariants $[\Omega, \Delta]$; q is a prime number not dividing $2\Omega\Delta$; m is an integer relatively prime to $2\Omega\Delta q$ and satisfying the conditions:

$$\text{the congruence } f(x,y,z) \equiv m \pmod{8\Omega\Delta} \text{ is solvable;} \tag{3}$$

$$\left(\frac{-\Delta m}{q}\right) = 1. \tag{4}$$

Denote by $t(f,m)$ the number of primitive representations of the number m by the form f. Then there exist constants m_0, $\varkappa > 0$, $\varkappa' > 0$, depending only on Ω, Δ, q, such that for $m \geq m_0$

$$\varkappa h(-\Delta m) < t(f,m) < \varkappa' h(-\Delta m). \tag{5}$$

It follows at once from this theorem that every sufficiently large integer m is representable by the form $f(x,y,z)$, provided it satisfies the necessary genus condition (3), as well as the additional condition (4), connected with the method of proof. Condition (4) can be eliminated by assuming the validity of the Riemann hypothesis, or certain weaker density hypotheses, for Dirichlet L-series.

Theorem 3. Let $f(x,y,z)$ be an integral primitive positive ternary quadratic form of odd relatively prime invariants $[\Omega, \Delta]$; q a prime number not dividing 2Δ; g an odd number relatively prime to $\Omega\Delta$; $\mathfrak{C}$ a conical region with vertex at the origin and lower (in the sense of Jordan) solid angle $\lambda > 0$. Consider an integer m, relatively prime to $2\Omega\Delta qg$, and integers x_0, y_0, z_0 satisfying the conditions:

$$f(x_0, y_0, z_0) \equiv m \pmod{8\Omega\Delta g}, \tag{6}$$

$$\left(\frac{-\Delta m}{q}\right) = 1. \tag{7}$$

Denote by $t(f, g, \mathfrak{C}; m)$ the number of those primitive representations (x, y, z) of the number m by the form f which lie in the cone $\mathfrak{C}$ and are congruent to (x_0, y_0, z_0) modulo g. Then there exist constants m_0, $\varkappa > 0$, $\varkappa' > 0$, depending only on Ω, Δ, q, g, and $\mathfrak{C}$, such that for $m \geq m_0$

$$\varkappa h(-\Delta m) < t(f, g, \mathfrak{C}; m) < \varkappa' h(-\Delta m). \tag{8}$$

Let us note that (as follows from the proof of the theorem), if one restricts oneself to convex conical regions $\mathfrak{C}$, then one may allow the convex region $\mathfrak{C}$ to depend on m, bounding from below only its solid angle $\lambda > 0$. The cases $g = 1$ or $\lambda = 4\pi$ are not excluded, so that Theorem 2 is a very special case of this theorem. On the other hand, our theorem, by enlarging the class of forms f under consideration, also includes the results of the paper [9]. Concerning condition (7) one may repeat the same thing that we said about condition (4). In some cases, however, this condition is contained in condition (6). Namely, from Theorem 3 it follows directly:

Theorem 4. Let $f(x, y, z)$ be an integral primitive positive ternary quadratic form of odd relatively prime invariants $[\Omega, \Delta]$, among whose genus characters there is the character

$$\left(\frac{-\Delta f}{q}\right) = 1,$$

where q is some prime number dividing Ω; g is an odd number relatively prime to $\Omega\Delta$; $\mathfrak{C}$ is a conical region with vertex at the origin and lower solid angle $\lambda > 0$. Consider an integer m, relatively prime to $2\Omega\Delta g$, and integers x_0, y_0, z_0 satisfying the condition

$$f(x_0, y_0, z_0) \equiv m \pmod{8\Omega\Delta g}. \tag{9}$$

Denote by $t(f, g, \mathfrak{C}; m)$ the number of those primitive representations (x, y, z) of the number m by the form f which lie in the cone $\mathfrak{C}$ and are congruent to (x_0, y_0, z_0) modulo g. Then there exist constants m_0, $\varkappa > 0$, and $\varkappa' > 0$, depending only on Ω, Δ, g, and $\mathfrak{C}$, such that for $m \geq m_0$

$$\varkappa h(-\Delta m) < t(f, g, \mathfrak{C}; m) < \varkappa' h(-\Delta m). \tag{10}$$

This theorem generalizes and sharpens the result of Yu. V. Linnik (^{7}) (the proof of which in (^{7}) is only sketched and differs substantially from ours). As usual, from the theorems formulated above on primitive representations of the number

m by the form f, one can derive the corresponding theorems on arbitrary integral representations. We formulate one of them.

Theorem 5. Let $f(x, y, z)$ be an integral primitive positive ternary quadratic form of odd relatively prime invariants $[\Omega, \Delta]$; let q be a prime number not dividing 2Δ; let g be an odd number relatively prime to $\Omega\Delta$; let $\mathfrak{C}$ be a conical region with vertex at the origin and lower solid angle $\lambda > 0$. Consider an integer m, relatively prime to $2\Omega\Delta qg$, and integers x_0, y_0, z_0 satisfying the conditions

$$f(x_0, y_0, z_0) \equiv m \pmod{8\Omega\Delta g}, \tag{11}$$

$$\left(\frac{-\Delta m}{q}\right) = 1. \tag{12}$$

Denote by $T(f, g, \mathfrak{C}; m)$ the number of those integral representations (x, y, z) of the number m by the form f which lie in the cone $\mathfrak{C}$ and are congruent to (x_0, y_0, z_0) modulo g. Then there exist constants $m_0, \varkappa > 0, \varkappa' > 0$, depending only on Ω, Δ, q, g, and $\mathfrak{C}$, such that for $m \geq m_0$

$$\varkappa H(-\Delta m) < T(f, g, \mathfrak{C}; m) < \varkappa' H(-\Delta m), \tag{13}$$

where $H(-\Delta m)$ is the number of all classes of positive binary integral quadratic forms of determinant Δm.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.