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Abstract

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ON BOUNDARY-VALUE PROBLEMS FOR GENERAL DIFFERENTIAL OPERATORS IN PARTIAL DERIVATIVES

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Recently there have appeared works in which, to one degree or another, boundary-value problems and expansions in eigenfunctions for nonelliptic equations are studied ([1–11,16]). In this note we shall consider the question of the solvability of boundary-value problems for the indicated equations in the class of certain generalized solutions. Let us note at once that the existence of such a generalized solution does not yet completely solve the problem in the usually understood sense. Namely, the study of the solvability of a boundary-value problem, generally speaking, splits into two parts: 1) the investigation of the density of the range of the corresponding operator in the space of right-hand sides, and 2) the investigation of one or another continuity property of the inverse operator. In this note we essentially touch only upon the first of these two questions. The constructions will be carried out for an equation of second order and for boundary conditions either homogeneous or removed; the extension to more general problems (including systems of equations) presents no essential difficulties.

1°. Slightly modifying in form the construction of Lax ([12]), we introduce the notion of a negative norm. Let G be a finite domain of the n-dimensional space E_n, bounded by a piecewise-smooth (i.e. piecewise twice continuously differentiable) boundary Γ; W_2^l is the Sobolev space of complex functions on G having square-summable l-th derivatives; $(u, v)_l$ is the scalar product in it. Since $\|u\|_0 \leqslant \|u\|_l$ ($W_2^0 = L_2$), the bilinear functional $(f, u)_0$ ($f \in L_2,\ u \in W_2^l$) is continuous with respect to variation of f in L_2 and u in W_2^l. Therefore it admits the representation $(f, u)_0 = (If, u)_l$, where I is a linear operator acting continuously from all of L_2 into W_2^l. Put, for $f, g \in L_2$, $(f, g)_{-l} = (If, g)_0 \equiv (If, Ig)_l$; the completion of L_2 with respect to this scalar product will be denoted by W_2^{-l}, the norm in W_2^{-l} is negative; $\|f\|_{-l} \leqslant \|f\|_0$. Obviously the operator I acts in the space W_2^l continuously with respect to this norm; therefore it can be extended to an operator I, acting continuously from all of W_2^{-l} into W_2^l. From the definition of W_2^{-l} it follows that I^{-1} is continuous and defined on all of W_2^l. Passing to the limit in the relations defining $(f, g)_{-l}$, we obtain that for any $\alpha, \beta \in W_2^{-l}$ and $f \in L_2$

$$(\alpha, f)_{-l} = (I\alpha, f)_0, \qquad (\alpha, \beta)_{-l} = (I\alpha, I\beta)_l. \tag{1}$$

2°. Consider the differential expression of the 2nd order

$$\mathcal{L}[u] = \sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j} + a(x)u \qquad (x \in G), \tag{2}$$

where $a_j(x)$ and $a(x)$ are complex and bounded in G; $a_{jk}(x)$ are real and have bounded first derivatives in G. With $\mathcal{L}$, for $u, v \in W_2^2$, construct the scalar product $(u, v)_{\mathcal{L}} = (\mathcal{L}[u], \mathcal{L}[v])_{-2}$; after identi-

elements u for which $(u, u)_{\mathcal{L}} = 0$, and by completing it we obtain a Hilbert space $H_{\mathcal{L}}$. Let us note that for $u \in W_2^2$, $\|u\|_{\mathcal{L}} \leq C\|u\|_2$. The operator $u \to \mathcal{L}[u]$ ($u \in W_2^2$) is continuous from $H_{\mathcal{L}}$ into W_2^{-2}; extending it by continuity, we obtain a continuous operator L from $H_{\mathcal{L}}$ into W_2^{-2}. On the range $\mathfrak{R}(L)$ the inverse operator L^{-1} is defined and continuous.

Let D be some piece of Γ (i.e. a finite number of domains on Γ having piecewise smooth boundary). By $\overset{\circ}{W}{}_2^2(\Gamma \setminus D)$ we denote the closure in the W_2^2-norm of the set of all functions on G, twice continuously differentiable up to the boundary, which vanish on $\Gamma \setminus D$, and by $\overset{\circ}{H}_{\mathcal{L}}(\Gamma \setminus D)$ the closure of $\overset{\circ}{W}{}_2^2(\Gamma \setminus D)$ in the $H_{\mathcal{L}}$-norm.

Let $f \in L_2$; we shall say that the boundary-value problem $\mathcal{L}[u] = f$, $u|_{\Gamma \setminus D} = 0$, $u|_D$ free, has a generalized solution φ, if there exists an element $\varphi \in \overset{\circ}{H}_{\mathcal{L}}(\Gamma \setminus D)$ such that $L\varphi = f$. Thus, in essence, solvability of the boundary-value problem in the generalized sense means the existence of a sequence $u_n \in \overset{\circ}{W}{}_2^2(\Gamma \setminus D)$ for which $\mathcal{L}[u_n] \to f$ in the sense of W_2^{-2}. If convergence in $\overset{\circ}{H}_{\mathcal{L}}(\Gamma \setminus D)$ implies convergence in some W_2^1, then $\varphi \in W_2^1$ and therefore is a "more classical" solution. In this note we shall not dwell on the question of the nature of the elements of $\overset{\circ}{H}_{\mathcal{L}}(\Gamma \setminus D)$ (with the exception of the elliptic case; see below).

3°. Let $A(x) = \|a_{jk}(x)\|$, and let $\nu(x)$ be the unit vector of the exterior normal to Γ. We shall assume that on Γ there may be situated a piece X_0 on which $(A(x)\nu(x), \nu(x)) = 0$ and $A(x)\nu(x) \neq 0$ (a piece of characteristics of zero order), and a piece X_1 on which $A(x)\nu(x) = 0$ (a piece of characteristics of first order). On the part of the boundary $\Gamma \setminus (X_0 \cup X_1)$ there may also be points at which $(A(x)\nu(x), \nu(x)) = 0$, but we assume that the surface measure of the set of such points is zero. Almost everywhere on $\Gamma \setminus X_1$ the unit conormal

$$\mu(x) = A(x)\nu(x)/|A(x)\nu(x)|$$

is defined. For points of X_0 the conormal is tangent; therefore the derivative $\partial u/\partial \mu$ is defined for smooth functions given only on X_0. For such functions we introduce the first-order differential expression

$$\mathfrak{M}[u] = |A(x)\nu(x)|\,\partial u/\partial\mu,$$

and let $\mathfrak{M}^+$ be the expression formally adjoint to $\mathfrak{M}$.

Theorem. In order that the boundary-value problem $\mathcal{L}[u] = f$, $u|_{\Gamma\setminus D} = 0$, $u|_D$ free, have a generalized solution for every $f \in L_2$, it is necessary and sufficient that every classical (i.e. belonging to W_2^2) solution v of the formally adjoint homogeneous equation $\mathcal{L}^+[v] = 0$, satisfying the adjoint boundary conditions, be equal to zero.

The adjoint boundary conditions have the form:

$$v|_{\Gamma\setminus(X_0\cup X_1)} = 0;$$

$$\partial v/\partial\mu\,\big|_{D\setminus(X_0\cup X_1)} = 0; \qquad v|_{(X_0\cup X_1)\setminus D} \text{ free}, \quad \alpha v|_{X_1\cap D} = 0,$$

where

$$\alpha(x) = \sum_j \left(a_j - \sum_k \partial a_{jk}/\partial x_k\right);$$

$$(\mathfrak{M}^+[v] - \mathfrak{M}[v] + \alpha v)\big|_{X_0\cap D} = 0.$$

Here we assume that $X_0 \cap D$ is situated on the smooth part of Γ and is at a positive distance from X_1.

Let us note that solvability of the problem for every $f \in L_2$ is equivalent to its solvability for every $f \in W_2^{-2}$.

We obtain the proof of the theorem in the following way. Suppose its conditions are fulfilled. Since L realizes a homeomorphism between $H_{\mathcal{L}}$ and $\mathfrak{R}(L) \subseteq W_2^{-2}$, the image of $\mathring{H}_{\mathcal{L}}(\Gamma\setminus D)$ is closed. Therefore, if

$$L(\mathring{H}_{\mathcal{L}}(\Gamma\setminus D)) \subset W_2^{-2},$$

then in W_2^{-2} there exists a vector $\alpha \perp L(\overset{0}{H}_{\mathcal{L}}(\Gamma\setminus D))$, i.e., in particular,

$$(\alpha, L[u])_{-2} = 0, \quad u \in \overset{0}{W}{}_2^2(\Gamma\setminus D).$$

Applying (1) to this equality, we conclude that the function $v = \overline{I\alpha} \in W_2^2$ is such that

$$(\overline{v}, \mathcal{L}[u])_0 = (I\alpha, \mathcal{L}[u])_0 = (\alpha, \mathcal{L}[u])_{-2} = 0$$

for all $u \in \overset{0}{W}{}_2^2(\Gamma \setminus D)$. In particular, taking here u that vanish in a strip near Γ and transposing $\mathcal{L}$ by Green' s formula, we find, by the arbitrariness of u, that $\mathcal{L}^+[v] = 0$. Further, by Green' s formula, for any $u \in \overset{0}{W}{}_2^2(\Gamma \setminus D)$ we have

$$\int_{\Gamma\setminus X_1} |A(x)|\, v(x) \left\{ \frac{\partial u}{\partial \mu} v - u \frac{\partial v}{\partial \mu} \right\} dx + \int_{\Gamma} \alpha(x) uv \, dx =$$

$$= (\mathcal{L}[u], \overline{v})_0 - (u, \overline{\mathcal{L}^+[v]})_0 = 0.$$

In view of the arbitrariness of u, it follows from this that v satisfies the adjoint boundary conditions. But then, by the assumption of the theorem, $v = \overline{I\alpha} = 0$, i.e. $\alpha = 0$. This proves sufficiency. Necessity is established, in essence, by reversing the above argument.

4°. Let us consider some examples.

1) Let G be a bounded domain with smooth boundary; let $\mathcal{L}$ be an elliptic expression with real coefficients and with a sufficiently large positive $a(x)$ so that

$$(\mathcal{L}[u], u)_0 \geq \varepsilon (u, u)_0 \quad (u \in \overset{0}{W}{}_2^2(\Gamma);\ \overline{u(x)} = u(x);\ \varepsilon > 0).$$

Pose the problem with the boundary condition $u|_\Gamma = 0$. In this case $D = X_0 = X_1 = 0$, and the adjoint boundary condition has the form $v|_\Gamma = 0$. The existence of a generalized solution φ follows immediately from the positivity of $\mathcal{L}$, and hence also of $\mathcal{L}^+$, on the functions indicated above. As is known ([13]), the problem under consideration has a solution $u \in \overset{0}{W}{}_2^2(\Gamma)$; by uniqueness of the generalized solution, $\varphi = u$ in $H_\mathcal{L}$. It is not difficult to show that in our case

$$\|u\|_\mathcal{L} \geq \delta \|u\|_0 \quad (u \in \overset{0}{W}{}_2^2(\Gamma),\ \delta > 0),$$

and therefore $\overset{0}{H}_\mathcal{L}(\Gamma)$ contains no ideal elements at all and coincides with $\overset{0}{W}{}_2^2(\Gamma)$.

2) In general, let $\mathcal{L}$ be an arbitrary expression of the form (2), which is known to be elliptic and to have sufficiently smooth coefficients in some subdomain $Q \subset G$ with piecewise smooth boundary. Then the generalized solution φ of the boundary-value problem posed in 2° coincides in Q with the classical one in the following sense: let $u_n \in \overset{0}{W}{}_2^2(\Gamma \setminus D)$ converge in $H_\mathcal{L}$ to φ; then there exists a sequence of functions $h_n \in L_2$, $h_n \perp u_n - h_n$, which inside Q belong to W_2^2 and satisfy the equation $\mathcal{L}[h_n] = 0$ $(x \in Q)$, such that $u_n - h_n$ converge in L_2 to some function u that belongs to

W_2^2 inside Q and satisfies the equation $\mathcal{L}[u] = f$ $(x \in Q)$. If Q adjoins some part Γ_1 of the boundary of the domain G on which zero boundary conditions were prescribed, then

$$h_n|_{\Gamma_1} = u|_{\Gamma_1} = 0.$$

3) Let $\mathcal{L}$ be an elliptic expression with arbitrary sufficiently smooth coefficients. Then the boundary-value problem

$$\mathcal{L}[u] = f, \quad u|_{\Gamma \setminus D} = 0,$$

with $u|_D$ omitted, always has a generalized solution. This follows at once from the theorem and from the uniqueness of the Cauchy problem for $\mathcal{L}$ ([14]). By 2), this solution is smooth outside D.

4) Let $\mathcal{L}$ be an arbitrary formally self-adjoint expression for which $\operatorname{Im} a(x)$ in G does not change sign and is almost everywhere nonzero; moreover, suppose that $X_0 = 0$. The boundary-value problem with boundary condition $u|_\Gamma = 0$ always has a generalized solution φ. Indeed, by virtue of

of the theorem it is necessary to verify that the solution $v \in W_2^2$ of the problem $\mathcal{L}[v] = 0$, $v|_{\Gamma \setminus X_1} = 0$, $v|_{X_1}$ removed, is zero. This follows from the equality $(\mathcal{L} - a)[v] = -av$, the reality and self-adjointness of the expression $\mathcal{L} - a$, and Green' s formula. In view of 2), the solution φ is classical in the regions of ellipticity of $\mathcal{L}$.

5) Let

$$\mathfrak{R}[u] = \sum_j b_j(x)\, \partial u / \partial x_j + b(x) u,$$

where b_j and b are real and have bounded first derivatives. Put $\mathcal{L}[u] = \mathfrak{R}^* \mathfrak{R}[u] + c(x)u$, where $c(x)$ is nonnegative and sufficiently large. Let G be an arbitrary domain. The boundary-value problem with boundary condition $u|_\Gamma = 0$ always has a generalized solution. This is easily verified on the basis of the theorem and the fact that $\mathcal{L}$ is sufficiently positive.

6) Consider the Tricomi differential expression

$$\mathcal{L}[u] = x_2 \partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2$$

in an arbitrary domain G with a piecewise-smooth boundary on which there are no pieces of characteristics from the hyperbolic half-plane $x_2 < 0$. The boundary-value problem

$$\mathcal{L}[u] + zu = f \in L_2, \qquad u|_\Gamma = 0,$$

where $z(x)$ is an arbitrary complex addition (i.e., a function of the form $a(x)$ in 4)), always has a generalized solution φ. This follows from 4), since $\mathcal{L} + z$ is formally self-adjoint. From 2) it follows that in any case, for $x_2 > 0$, φ is classical.

7) Consider the usual formulation of the boundary-value problem for the Tricomi equation: G is bounded by a piecewise-smooth curve Γ_1 for $x_2 \geq 0$ and by two pieces Γ_2 and Γ_3 (the left and right ones) of characteristics of order 0 for $x_2 \leq 0$; the equation $\mathcal{L}[u] = f \in L_2$ is considered with boundary conditions

$$u|_{\Gamma_1 \cup \Gamma_2} = 0, \qquad u|_{\Gamma_3} \text{ removed.}$$

According to the theorem, for solvability of this problem it is necessary to show that if $v \in W_2^2$ satisfies the equation $\mathcal{L}[v] = 0$ and the boundary conditions

$$v|_{\Gamma_1} = 0, \qquad v|_{\Gamma_2} \text{ removed}, \qquad \mathfrak{M}^*[v] - \mathfrak{M}[v]\big|_{\Gamma_2} = 0,$$

then $v = 0$. The equation $(\mathfrak{M}^* - \mathfrak{M})[v] = 0$ is of first order on Γ_3, singular at the point x_0 where Γ_3 and Γ_1 meet; $v(x_0) = 0$, since $v|_{\Gamma_1} = 0$. It is easy to write it out and to show that, despite the singularity, the Cauchy condition $v(x_0) = 0$ entails the equality $v|_{\Gamma_3} = 0$. Thus,

$$v|_{\Gamma_1 \cup \Gamma_2} = 0, \qquad v|_{\Gamma_3} \text{ removed.}$$

From the uniqueness theorem for the Tricomi problem [15] it follows that $v = 0$. Thus our problem has a generalized solution, which in view of 2) is classical for $x_2 > 0$. From the classical investigations of the Tricomi problem, as is known, its sufficient smoothness also follows for $x_2 \leq 0$.

8) Let $\mathcal{L}$ be a formally self-adjoint expression with real coefficients; suppose that $X_0 = 0$. The operator in L_2

$$\Lambda g = \mathcal{L}[g], \qquad g \in \mathfrak{D}(\Lambda) = \mathring{W}_2^2(\Gamma),$$

is obviously Hermitian. In some cases one can show that its closure is self-adjoint. Then the equation

$$\overline{\Lambda} g - zg = f \in L_2, \qquad (z \notin \text{the spectrum})$$

has a solution $g \in \mathfrak{D}(\overline{\Lambda})$, which is, obviously, also a generalized solution of the corresponding boundary-value problem. We shall not dwell on these questions in greater detail here.

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REFERENCES

1. L. Hörmander, *Acta Math.*, **94**, No. 3-4 (1955).

2. S. L. Sobolev, DAN, **109**, No. 4 (1956).

3. N. N. Vakhaniya, DAN, **116**, No. 6 (1957).

4. D. I. Mangeron, *Bul. Inst. Politechn. Iasi*, **4** (1949).

5. F. Manaresi, *Rend. Seminar mat. Univ. Padova*, **23**, No. 1 (1954).

6. I. S. Louhivaara, *Rend. Circ. Mat. Palermo*, ser. II, **5** (1956).

7. W. Littman, *Comm. Pure and Appl. Math.*, **11**, No. 1 (1958).

8. M. I. Vishik, L. A. Lyusternik, *Usp. matem. nauk*, **12**, No. 5 (1957).

9. M. I. Vishik, A. G. Kostyuchenko, DAN, **103**, No. 3 (1955).

10. Yu. M. Berezanskii, *Matem. sborn.*, **43(85)**, No. 1 (1957).

11. G. I. Kats, DAN, **119**, No. 1 (1958).

12. P. D. Lax, *Comm. Pure and Appl. Math.*, **8**, No. 4 (1955) (see also the collection of translations *Matematika*, **1**, No. 1 (1957)).

13. O. A. Ladyzhenskaya, *Vestn. LGU*, No. 11 (1955).

14. E. M. Landis, DAN, **107**, No. 5 (1956).

15. F. Tricomi, *Lectures on Partial Differential Equations*, IL, 1957.

16. M. I. Vishik, S. L. Sobolev, DAN, **111**, No. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.