HYDROMECHANICS

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Abstract

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HYDROMECHANICS

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ON THE THEORY OF GAS JETS

1°. The S. A. Chaplygin equation for determining the stream function $\psi(\theta, \tau)$ of a steady gas flow,

$$\frac{\partial}{\partial \tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial \psi}{\partial \tau}\right\}+\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2 \psi}{\partial \theta^2}=0 \tag{1}$$

possesses, in addition to those particular solutions which were indicated and used by S. A. Chaplygin, one more family of particular solutions, making it possible to solve a number of new problems on the jet motion of a gas.

This family consists of functions that are the product of a function $\Theta(\theta)$ by a function $T(\tau)$, where these functions are integrals of the differential equations

$$\begin{gathered}\frac{d^2\Theta}{d\theta^2}-n^2\Theta=0,\\ \frac{d}{d\tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{dT}{d\tau}\right\}+n^2\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}T=0,\end{gathered} \tag{2}$$

in which n is an arbitrary constant number. Two linearly independent particular solutions of equation (2) can be represented in the following form:

$$\begin{aligned}T_n' &= M(\tau,n)\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)-N(\tau,n)\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right),\\ T_n'' &= M(\tau,n)\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)+N(\tau,n)\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right),\end{aligned} \tag{3}$$

where τ_2 is some number, and the functions M and N are defined by the series

$$\begin{aligned}M(\tau,n) &= 1+\frac{\beta}{2}\frac{n^2(1-\frac{1}{2}\beta n^2)}{n^2+4}\tau^2+\cdots,\\ N(\tau,n) &= \left[-\frac{1}{2}\beta\tau-\frac{1}{2}\beta\frac{(1+\frac{1}{2}\beta)n^2-2(\beta+1)}{n^2+4}\tau^2+\cdots\right]n.\end{aligned} \tag{4}$$

Let us note that, in solving the simplest Sturm–Liouville problem for equation (3) with the conditions $T = 0$ for two values of τ smaller than

$$\frac{1}{2\beta + 1},$$

the corresponding fundamental number $\lambda = n^2$ will always be positive.

We transform equation (3) to new variables z and $u(z)$, putting

$$z = \int_{\tau}^{\tau_2} \frac{d\tau}{2\tau} \sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}}; \qquad u(z) = T(\tau) \sqrt[4]{\frac{1 - (2\beta + 1)\tau}{(1 - \tau)^{2\beta+1}}};$$

we obtain:

$$\frac{d^2u}{dz^2} + \left[n^2 + \frac{\beta(2\beta + 1)\tau^2}{1 - \tau} \frac{4 - 2(\beta + 2)\tau - \beta(2\beta + 1)\tau^2}{[1 - (2\beta + 1)\tau]^3} \right] u = 0. \tag{5}$$

We shall construct for this differential equation the inversion formulas, starting from its particular solution $\omega(z, \lambda)$, which becomes equal to,

zero at $z = 0$ and considered for all positive values of z from zero to ∞. In view of the above-noted property of the number n^2 and the convergence of the integral

$$\int_0^{\infty} \frac{\beta(2\beta + 1)\tau^2}{1 - \tau} \frac{4 - 2(\beta + 2)\tau - \beta(2\beta + 1)\tau^2}{[1 - (2\beta + 1)\tau]^3} \, dz$$

equation (5) has a continuous spectrum located on the positive part of the λ-axis ([1]).

The indicated inversion formulas, which are constructed using the expansions (3) and (4), have the form

$$F(\lambda) = \int_0^{\infty} f(x)\, \omega(x, \lambda)\, dx, \qquad f(x) = \int_{-\infty}^{\infty} \omega(x, \lambda) F(\lambda)\, d\rho(\lambda),$$

where

$$\omega(x, \lambda) = \frac{4\tau_2^2}{\sqrt{\lambda}} \sqrt[4]{\frac{1 - (2\beta + 1)\tau}{[1 - (2\beta + 1)\tau_2]^3} \frac{(1 - \tau_2)^{3-2\beta}}{(1 - \tau)^{2\beta+1}}}\, T(\tau, n);$$

$$\lambda = n^2; \qquad T(\tau, n) = T_n'(\tau_2) T_n''(\tau) - T_n''(\tau_2) T_n'(\tau);$$

Fig. 1

Figure 1: Fig. 1

$$d\rho(\lambda) = \frac{1}{16\pi\tau_2^2}\sqrt[4]{\frac{[1-(2\beta+1)\tau_2]^3}{(1-\tau_2)^{3-2\beta}}}\,\frac{\sqrt{\lambda}\,d\lambda}{M(\tau_2,n)^2+N(\tau_2,n)^2}.$$

2°. Consider a vessel bounded by two parallel walls extending to infinity in one direction, and provided with a nozzle consisting of two small straight-line segments, equal to one another, inclined to the centerline of the vessel and departing from the free ends of the indicated parallel walls (Fig. 1). From such a vessel gas under pressure issues into free space in the form of a jet. Our problem consists in constructing the stream function of this gas flow.

Fig. 1

Let τ_1 and τ_2 denote the values of Chaplygin' s variables τ, respectively, in the remote parts of the vessel and on the jet. Further, let θ_0 denote the angle of inclination of the straight-line segments to the centerline of the vessel; let q be the value of the stream function along the centerline of the vessel, if the value of this function on the line $ABCD$ is taken to be zero.

To construct the stream function of the gas flow under consideration, we take the solution of Chaplygin' s equation indicated in 1° in the form

$$\psi = T_n(\tau)\frac{\operatorname{sh} n(\theta_0-\theta)}{\operatorname{sh} n\theta_0}$$

and apply the inversion formulas to it. We then obtain, after carrying out a number of operations, the following result:

$$\psi = \frac{2q}{\pi}\int_0^\infty \left[\frac{2\tau}{(1-\tau)^\beta}\frac{dT}{d\tau}\right]_{\tau_1}^{\tau_2}\frac{\operatorname{sh} n(\theta_0-\theta)}{n^2\operatorname{sh} n\theta_0}\,\frac{T(\tau,n)\,dn}{M(\tau_2,n)^2+N(\tau_2,n)^2}.$$

With the aid of similar calculations, a number of other problems in the theory of gas jets, considered for an incompressible fluid by N. E. Zhukovsky, can also be solved.

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CITED LITERATURE

1. B. M. Levitan, *Expansions in Eigenfunctions of Second-Order Differential Equations*, chs. II, III, Moscow—Leningrad, 1950.