MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

E. V. VORONOVSKAYA

ON CHEBYSHEV APPROXIMATION OF ANALYTIC FUNCTIONS BY ALGEBRAIC POLYNOMIALS

(Presented by Academician S. L. Sobolev on 4 III 1958)

Let

$$f(x)=\sum_{0}^{\infty}\alpha_i x^i$$

for any point of the segment $[0,1]$. It is required to find $Y_h(x)$ of degree not exceeding h, satisfying the condition:

$$\max_{[0,1]}|f(x)-Y_h(x)|=L\ (\min).$$

Let all points of deviation form, with account taken of the sign, a distribution $(\sigma_i^{\pm})_1^s$; we shall call it the f_h-distribution. It is evident that, for an unknown $f(x)$, this distribution may be any prescribed finite one, with the sole condition that it satisfy the requirement of the Chebyshev alternation. It is also evident that one may restrict oneself to considering functions of the form

$$f(x)=\sum_{h+1}^{\infty}\alpha_i x^i.$$

In all that follows we shall assume that the extremal polynomials $\{Q(x)\}$ of functionals of class II are known [(1)]. Recall that the resolvent of $Q(x)$

$$R_s^-(x)=\prod_{1}^{s}(x-\sigma_i^{\pm}),$$

and if

$$R_{s_1}^-(x)=\prod(x-\sigma_i^{+}),\qquad R_{s_2}^-(x)=\prod(x-\sigma_i^{-}),$$

then

$$Q(x) = 1 - R_{s_1}^2(x)\Phi(x) = -1 + R_{s_2}^2(x)\Psi(x),$$

where $\Phi(x) \geq 0$ and $\Psi(x) \geq 0$ on $[0, 1]$, if $(\sigma_i^{\pm})_1^s$ is a subdistribution of $Q(x)$.

Theorem 1. A. Every extremal polynomial of class I or II $Q(x)$ of the f_h-distribution $(\sigma_i^{\pm})_1^s$ gives for the function $f(x)$ the unique decomposition

$$f(x) = Y_h(x) + LQ(x) + LR_s^2(x)\Omega_Q(x), \tag{1}$$

where $R_s^2(x)$ is the square of the resolvent (with the corresponding variants [(2)]), and $\Omega_Q(x)$ is an analytic function.

B. If we have an identical representation of the given $f(x)$:

$$f(x) = Z_h(x) + MQ_p(x) + MR_s^2(x)\Omega(x) \qquad (h < p), \tag{2}$$

where $Q_p(x)$ is a polynomial of class I or II with a subdistribution containing not fewer than $h+1$ changes of sign, and $R_s^2(x)$ is the square of the resolvent of this subdistribution, while $\Omega(x)$ is an analytic function which, for $0 \leq x \leq 1$, does not go outside a certain interval (4), then $Z_h(x)$ is the polynomial of least deviation from $f(x)$, and this deviation is $L = M$.

Before the proof we make some remarks. Let $(\sigma_i)_1^s$ be any s points on $[0, 1]$. We shall call the remainder on division of $f(x)$ by a polynomial of the form

$$R_s^2(x) = \prod_1^s (x - \sigma_i)^2$$

the polynomial $r(x)$ of degree less than $2s$, obtained—

giving the successive expansion: 1) of $f(x)$ in a Taylor series at the point σ_1; 2) of the incomplete quotient $f_1(x)$ at the point σ_2, etc., namely: $f(x) = f(\sigma_1) + f'(\sigma_1)(x - \sigma_1) + (x - \sigma_1)^2 f_1(x) = f(\sigma_1) + f'(\sigma_1)(x - \sigma_1) + f_1(\sigma_2)(x - \sigma_1)^2 + f_1'(\sigma_2)(x - \sigma_1)^2(x - \sigma_2) + (x - \sigma_1)^2(x - \sigma_2)^2 f_2(x) = \cdots$, after which

$$f(x) = r(x) + R_s^2(x)\Omega(x). \tag{3}$$

To guarantee uniqueness when passing from one Taylor expansion to another, it is quite sufficient to assume, for example, that $f(x)$ is analytic inside two circles C_0 and C_1 of radius greater than 1 with centers at the points 0 and 1; then $\Omega(x)$—the incomplete quotient—is analytic in the same domain. In what follows we shall consider only such $f(x)$.

It is very easy to verify that the expansion (3) does not change under a change in the numbering of the points (σ) and, in general, is unique for the given $R_s^2(x)$. Let us define that $f(x)$ is divisible by $R_s^2(x)$ if and only if $r(x) \equiv 0$. Then the necessary and sufficient condition for divisibility is: σ_i is a zero of $f(x)$ of multiplicity not less than 2.

We return to Theorem 1. According to the condition

$$\max_{[0,1]} \frac{|f(x) - Y_h(x)|}{L} = 1 \tag{*}$$

and it is attained at all points $(\sigma_i)_1^s$ of the subdistribution $Q(x)$, and only at them. Then for

$$F(x) = \frac{f(x) - Y_h(x)}{L} - Q(x)$$

we have $F(\sigma_i) = 0$ and $F'(\sigma_i) = 0$ for the points (σ) internal to $[0,1]$. Thus, $f(x) - Y_h(x) - LQ(x)$ is divisible by R_s^2. Assertion A is proved, and $\Omega(x)$ does not go outside the interval

$$\left(-\frac{\Psi(x)}{R_{s_1}^2(x)}, +\frac{\Phi(x)}{R_{s_2}^2(x)} \right), \qquad 0 \le x \le 1, \tag{4}$$

which follows from condition (*) and from the structure of $Q(x)$. Conversely, if (2) holds, then $f(x) - Z_h(x)$ takes the values $\pm M$ at the points $(\sigma_i)_1^s$, with $s \ge h+2$, and with a sufficient number of alternations. Moreover,

$$\max_{[0,1]} \frac{|f(x) - Z_h(x)|}{M} = 1.$$

Assertion B is true.

Theorem 2. If $f(x) = \sum_{n+1}^{\infty} \alpha_i x^i$ is given, $Y_h(x)$ is the polynomial of its best approximation on $[0,1]$ with deviation L, and $(\sigma_i)_1^s$ is the f_h-distribution, then among the extremal polynomials of the f_h-distribution one can always, for sufficiently large N, find such a

$$Q_{N+2s}(x) = \sum_0^{N+2s} \bar{q}_i x^i,$$

for which

$$L\bar{q}_i = \alpha_i \quad (i = h+1, \dots, N).$$

Indeed, let $Q_n(x)$ be the principal polynomial of $(\sigma_i)_1^s$, i.e. the polynomial of lowest degree. We have:

$$Q_n(x) = 1 - R_{s_1}^2(x)\varphi(x) = -1 + R_{s_2}^2(x)\psi(x)$$

with $\varphi(x) \geq 0$ and $\psi(x) \geq 0$ on $0 \leq x \leq 1$. According to (1),

$$f(x) = Y_h(x) + LQ_n(x) + R_s^2(x)L\Omega_n(x). \tag{5}$$

$$L\Omega_n(x) = \sum_0^\infty b_i x^i = \sum_0^N b_i x^i + \rho_N(x)$$

satisfies the inequalities

$$-\frac{L\psi(x)}{R_{s_1}^2(x)} < L\Omega_n(x) < \frac{L\varphi(x)}{R_{s_2}^2(x)}, \qquad 0 \leq x \leq 1, \tag{6}$$

and moreover it lies **inside** the indicated strip of width $2L/R_s^2(x)$, owing to the absence of other points of deviation. Choose N sufficiently large so that

$$S_N(x) = \sum_0^N b_i x^i$$

also lies **inside** the strip (6). Then

$$f(x) = Y_h(x) +$$

$$+L\left[Q_n(x) + R_s^2(x)\frac{1}{L}S_N(x)\right] + R_s^2(x)\rho_N(x) = Y_h(x) + LQ_{N+2s}^-(x) + R_s^2(x)\rho_N(x),$$

and the theorem is proved.

Corollary. $Y_h(x) = -L\sum_0^h \overline{q}_i x^i$.

Let us note that $Q_{N+2s}^-(x)$, for $N = N_{\min}$, is determined quite uniquely and belongs, generally speaking, to polynomials of class I.

Theorem 3. If $f(x)$ and $\varphi(x)$ are continuous and $\varphi(x) \geq 0$ on $[a, b]$, and if $(\sigma_i)_1^s$ is the complete set of (distinct) zeros of $\varphi(x)$ on $[a, b]$, then the best approximation of $f(x)$ by polynomials of the form

$$\varphi(x)\sum_0^k c_i x^i$$

on the closed set $E(a,b)=E(\varepsilon)$ with the open intervals $(\sigma_i-\varepsilon,\sigma_i+\varepsilon)$ and $[a,a+\varepsilon)$, $(b-\varepsilon,b]$ removed is subject to Chebyshev's theorem, i.e. there exists a unique polynomial $\varphi(x)P_k(x,\varepsilon)$ which on $E(\varepsilon)$ deviates least from $f_{\pm}(x)$, and the points of deviation form on $E(\varepsilon)$ a distribution (ξ_i) with number of alternations $q\geq k+1$.

The proofs of necessity and sufficiency, as well as uniqueness, follow almost exactly the model given, for example, in S. N. Bernstein' s monograph ([3], Ch. I, § 5). A direct application of Chebyshev' s theorem here is impossible, since the component functions $\{\varphi(x)x^i\}$ do not form a T-system on $[a,b]$.

Theorem 4. If

$$f(x)=\sum_{0}^{N+2s} q_i x^i = Q^-_{N+2s}(x)$$

and $\varphi(x)=R_s^2(x)x^{N+1}$ (see Theorem 2), then for any $k\geq 1$ there exists a polynomial of the form

$$R_s^2(x)x^{N+1}\sum_{0}^{k} c_i x^i = R_s^2(x)x^{N+1}P,$$

which deviates least from $Q^-_{N+2s}(x)$ on $[0,1]$, and the number of points of deviation is $s\geq k+2$.

Indeed, on $E(0,1)=E(\varepsilon)$, with the points (σ) removed, there exists, by Theorem 3, a unique polynomial of best approximation of the form

$$R_s^2(x)x^{N+1}P_k(x,\varepsilon);$$

let this polynomial be

$$Q_m(x,\varepsilon)=Q^-_{N+2s}(x)-R_s^2(x)x^{N+1}P_k(x,\varepsilon)$$

with deviation $L(\varepsilon)$ and with not fewer than s points of deviation $(\xi_i)_1^s$. The question consists in the possibility of passing to the limit as $\varepsilon\to 0$. Note that $L(\varepsilon)<1$, since

$$L(\varepsilon)<\max_{[0,1]}|Q^-_{N+2s}(x)|=1$$

when $c_i=0$. Owing to the continuity of

$$\sum_{0}^{k} c_i(\varepsilon)x^i$$

we have $s(\varepsilon)\geq k+2$ for every ε; $L(\varepsilon)$, increasing, tends to 1 as $\varepsilon\to 0$, since at the points σ_i we have

$$Q^{-(\pm)}_{N+2s}(\sigma_i)=\pm 1.$$

The existence of

$$\lim_{\varepsilon \to 0} c_i(\varepsilon) = c_i$$

is proved on the basis of boundedness of the coefficients. Denote the limiting polynomial by

$Q_m(x)$; $\max |Q_m(x)| = 1$. The number of its nodes is $s \geq k+2$, and the number of alternations is $q \geq k+1$.[1] We note that, in the limit, the uniqueness theorem is violated.

Corollary. Among the polynomials of class II with h-distribution $(\sigma_i^{\pm})_1^{\bar{s}}$, for sufficiently large m there is a polynomial

$$Q_m^*(x) = \sum_0^m q_i x^i,$$

for which the coefficients $q_i = \bar{q}_i$ $(i = 0, 1, \dots, N)$.

This follows directly from the formula

$$Q_m(x) = Q_{N+2\bar{s}}(x) - R_s^2(x) x^{N+1} \sum_0^k c_i x^i, \tag{7}$$

valid for any $k > 0$, in which we take $k+2 > m/2+1$, which ensures that $Q_m^*(x)$ belongs to class II and gives a sufficient condition for this: $m > 2(N + 2\bar{s})$.

The theorems proved indicate ways of finding $Y_h(x)$ for a given

$$f(x) = \sum_{h+1}^{\infty} \alpha_i x^i.$$

Fixing a certain m, choose a passport $[m, s, q]$ (q is the number of sign alternations in the distribution), in which $q \geq h+1$, and take the polynomials $\{Q_m(x)\}$ of this passport, known by their analytic form. This family is of class II and contains l variable parameters, where $l = m + 1 - s$; they may be chosen as the coefficients of $Q_m(x)$ at $x^{h+1}, x^{h+2}, \dots, x^{h+l}$; denote them respectively by $\vartheta_{h+1}, \vartheta_{h+2}, \vartheta_{h+l}$; denote $h + l = N - 1$ and put $\vartheta_i = \alpha_i / M$ $(i = h+1, \dots, N-1)$. The remaining coefficients of the polynomial $Q_m(x)$ are certain functions of (ϑ_i); put

$$q_N = q_N(\alpha_{h+1}/M, \dots, \alpha_{N-1}/M) = \alpha_N^*/M;$$

from this dependence M is determined. The polynomial $Q_m(x)$ found in this way is the only one from the passport $[m, s, q]$ that could turn out to be the desired $Q_m^*(x)$ (by the corollary to theorem 4). The suitability of

$$Q_m(x) = \sum_0^m q_i x^i$$

[1] 0,1

is checked by means of the criterion following from theorem 1, namely: $Q_m(x)+Q_m^*(x)$ if and only if it contains the h-distribution as its subdistribution, and in this case

$$M = L, \qquad -M\sum_0^h q_i x^i = Y_h(x).$$

Let $(\rho_i^\pm)_1^s$ be the distribution of $Q_m(x)$ (complete). Then, according to (1),

$$f(x) - M\sum_{h+1}^m q_i x^i$$

must have, as its zeros of multiplicity not less than two, some subdistribution $(\sigma_i^\pm)_1^{\bar{s}}$ of $(\rho_i^\pm)_1^s$, containing at the same time no fewer than $h+1$ sign alternations. If this requirement is fulfilled and if $R_s^2(x)$ is the square of the resolvent of $(\sigma_i^\pm)_1^{\bar{s}}$, then in the identity

$$f(x) - M\sum_{h+1}^m q_i x^i = R_s^2(x) M\Omega(x)$$

the function $\Omega(x)$ must not go outside an interval of the form (4). Then $Q_m = Q_m^*$ and $Y_h(x)$ has been found. If, however, $Q_m(x)$ is not suitable, then one should seek Q_m among polynomials of another passport with the same m, and then increase m. After a finite number of trials, Q_m^*, and consequently also $Y_h(x)$, will be found.

Leningrad Institute
of Aviation Instrument Engineering

Received
7 II 1958

REFERENCES

1. E. V. Voronovskaya, DAN, 114, No. 5 (1957).
2. E. V. Voronovskaya, *Extremal polynomials of finite functionals*, Dissertation abstract, LGU, 1955.
3. S. N. Bernstein, *Extremal properties of polynomials*, 1937.

Note: Figure translations are in progress. See original paper for figures.