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MATHEMATICS

A. M. POLISUEV

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Abstract

Full Text

MATHEMATICS

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ON THE UNIFORM DISTRIBUTION OF A SYSTEM OF FUNCTIONS THAT IS A SOLUTION OF A SYSTEM OF LINEAR FINITE-DIFFERENCE EQUATIONS OF THE FIRST ORDER

(Presented by Academician I. M. Vinogradov on 28 VI 1958)

Let a system of functions $\varphi_1(x), \dots, \varphi_s(x)$ be a solution of the following system S of linear finite-difference equations of the first order with integer coefficients

$$\begin{aligned} \Phi_1(x+1) &= a_{11}\Phi_1(x) + \dots + a_{s1}\Phi_s(x), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Phi_s(x+1) &= a_{1s}\Phi_1(x) + \dots + a_{ss}\Phi_s(x). \end{aligned} \tag{1}$$

We assume that the determinant of the matrix

$$\begin{pmatrix} a_{11} & \cdot & \cdot & a_{s1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1s} & \cdot & \cdot & a_{ss} \end{pmatrix} \tag{2}$$

of the system (1) is different from zero.

Consider, in the unit hypercube,

$$0 \le x_1 \le 1, \dots, \quad 0 \le x_s \le 1 \tag{3}$$

an arbitrary domain v, whose volume we denote by $|v|$.

Let $N_p(v)$ be the number of points $M(\{\varphi_1(x)\}, \dots, \{\varphi_s(x)\})$ that fall into the domain v for $x = 1, 2, \dots, p$. The system of functions $\varphi_1(x), \dots, \varphi_s(x)$ is called **uniformly distributed** in s-dimensional space if

$$\lim_{p\to\infty} \frac{N_p(v)}{p} = |v|$$

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(see ([1])).

Theorem 1. *The system of functions $\varphi_1(x), \dots, \varphi_s(x)$, which is a solution of the system (1), is uniformly distributed in s-dimensional space if none of the roots of the characteristic polynomial of the matrix (2) is equal in modulus to one and if, for every hypercube of volume $|v|$ with sides parallel to the coordinate axes and lying entirely inside the unit hypercube (3), the relation*

$$\lim_{p\to\infty} \frac{N_p(v)}{p} < c|v|,$$

holds, where c is a certain constant.

For an exponential function, a similar theorem was previously proved by I. I. Shapiro-Pyateckii ([2]) and A. G. Postnikov ([3], [4]).

Using Theorem 1, one can prove the theorem formulated below, Theorem 2.

Let $X_1, \dots, X_s$ run through all integers. Then the points whose coordinates are equal to the corresponding coordinates of the vectors

$$(X_1, \dots, X_s) \begin{pmatrix} a_{11} \cdot \dots \cdot a_{1s} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ a_{s1} \cdot \dots \cdot a_{ss} \end{pmatrix}^{-k},$$

where k is a natural number, form in s-dimensional space a lattice of parallelepipeds. We shall call this lattice the lattice of rank k. Let $\Delta^{(k_1)}$ be one of the parallelepipeds of the lattice of rank k_1. Define the mapping of the hypercube (3) onto $\Delta^{(k_1)}$ as follows:

$$f_{\Delta^{(k_1)}}(x_1, \dots, x_s) =$$

$$= (X_1^{(k_1)}, \dots, X_s^{(k_1)}) \begin{pmatrix} a_{11} \cdot \dots \cdot a_{1s} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ a_{s1} \cdot \dots \cdot a_{ss} \end{pmatrix}^{-k_1} + (x_1, \dots, x_s) \begin{pmatrix} a_{11} \cdot \dots \cdot a_{1s} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ a_{s1} \cdot \dots \cdot a_{ss} \end{pmatrix}^{-k_1},$$

where $X_1^{(k_1)}, \dots, X_s^{(k_1)}$ are integers determined by the parallelepiped $\Delta^{(k_1)}$.

The image of some parallelepiped $\Delta^{(k_2)}$ of the lattice of rank k_2, lying wholly inside the hypercube (3), will be denoted by $\Delta^{(k_1)}\Delta^{(k_2)}$. It is clear that

$$\Delta^{(k_1)}\Delta^{(k_2)} \subset \Delta^{(k_1)}.$$

Next we define inductively

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$$\Delta^{(k_1)} \dots \Delta^{(k_n)} = (\Delta^{(k_1)} \dots \Delta^{(k_{n-1})})\Delta^{(k_n)}, \qquad n = 3, 4, \dots$$

Now take the parallelepipeds of ranks $1, 2, \dots$ lying wholly inside the hypercube (3), and number them arbitrarily within each rank:

$$\Delta_1^{(1)}, \dots, \Delta_i^{(1)}$$

$$. \ . \ . \ . \ .$$

$$\Delta_1^{(r)}, \dots, \Delta_{i_r}^{(r)}$$

$$. \ . \ . \ . \ .$$

Construct the sequence of closed sets nested one inside another

$$\Delta_1^{(1)},\ \Delta_1^{(1)}\Delta_2^{(1)}, \dots, \Delta_1^{(1)}\Delta_2^{(1)} \dots \Delta_i^{(1)},\ \Delta_1^{(1)}\Delta_2^{(1)} \dots \Delta_i^{(1)}\Delta_1^{(2)}, \dots$$

$$\dots, \Delta_1^{(1)}\Delta_2^{(1)} \dots \Delta_i^{(1)}\Delta_1^{(2)} \dots \Delta_{i_2}^{(2)}, \dots \tag{4}$$

Theorem 2. *Let all roots of the characteristic polynomial of the matrix (2) be, in modulus, greater than one. Then there exists a unique point $N(\mu_1, \dots, \mu_s)$ belonging to all the sets of the sequence (4); the system of functions $\varphi_1(x), \dots, \varphi_s(x)$, which is a solution of the system (1) with initial values $\varphi_1(1) = \alpha_1, \dots, \varphi_s(1) = \alpha_s$, where $\alpha_1 = \mu_1 a_{11} + \dots + \mu_{sa_{s1}}, \dots, \alpha_s = \mu_1 a_{1s} + \dots + \mu_{sa_{ss}}$, is uniformly distributed in s-dimensional space.*

We prove both theorems by the method of the paper ([4]).

Remark. From Theorem 2 it follows, in particular, that the system of functions $\alpha_1 q_1^x, \dots, \alpha_s q_s^x$, where $q_1, \dots, q_s$ are integers greater than one, is uniformly distributed. This result was obtained earlier in the paper ([5]) by another method.

From Theorem 2 there also follows the uniform distribution of the exponential function in a complex domain considered in the paper ([4]).

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REFERENCES

[1] H. Weyl, Math. Ann., **77**, 313 (1916).
[2] I. I. Shapiro-Pyatetskii, Izv. AN SSSR, ser. matem., **15**, 47 (1951).
[3] A. G. Postnikov, DAN, **86**, 473 (1952).
[4] A. G. Postnikov, Vestn. LGU, No. 13, issue 3, 81 (1957).
[5] N. M. Korobov, DAN, **84**, 13 (1952).

Note: Figure translations are in progress. See original paper for figures.

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