Mathematics

V. K. Saul’ ev

1958

SovietRxiv

Abstract

Full Text

Mathematics

V. K. Saul' ev

On Methods of Increased Accuracy and Two-Sided Approximations to the Solution of Parabolic Equations

(Presented by Academician S. L. Sobolev on 11 IX 1957)

1. Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad 0 < t \le T;$$

$$u(x,0) = f(x); \qquad u(0,t) = u(1,t) = 0. \tag{1}$$

We shall solve problem (1) by the method of finite differences, using the difference equation

$$-\alpha\left(v_{i-1,k+1} + v_{i+1,k+1}\right) + 2(\omega+\alpha)v_{i,k+1}$$

$$= (2-\alpha)\left(v_{i-1,k} + v_{i+1,k}\right) - 2(2-\omega-\alpha)v_{i,k}, \tag{2}$$

where $v_{i,k} = v(i\Delta x, k\Delta t)$; $\Delta x = \frac{1}{n+1}$; $\Delta t > 0$; $\omega = \frac{(\Delta x)^2}{\Delta t}$; $0 \le \alpha \le 1$.

Let the solution of problem (1) exist and have derivatives with respect to t, bounded in absolute value in $(0 < x < 1,\ 0 \le t \le T)$, up to the fourth order. Then the following theorem holds:

Theorem 1. *If the condition $\omega \ge 2(1-\alpha)$ is satisfied (the stability condition for equation (2)), the relations*

$$\sup_k \left[\frac{1}{n}\sum_{i=1}^{n}(u_{i,k} - v_{i,k})^2\right]^{1/2} = \begin{cases} O\left((\Delta x)^2\right), & \text{if } \alpha \ne 1-\omega/6,\ \alpha \ge 1-\omega/2; \\ O\left((\Delta x)^4\right), & \text{if } \alpha = 1-\omega/6,\ \omega \ne 2\sqrt{5}; \\ O\left((\Delta x)^6\right), & \text{if } \alpha = 1-\omega/6,\ \omega = 2\sqrt{5}. \end{cases}$$

2. Now consider the difference equation (1)

$$v_{i,k+1} = \frac{1}{2(1+\omega)^p}\left(v_{i-p,k+1} + v_{i+p,k+1}\right) - \frac{1-\omega}{2(1+\omega)^p}\left(v_{i-p+1,k} + v_{i+p-1,k}\right) +$$

$$+\frac{\omega^2}{2(1+\omega)^2}\sum_{\gamma=0}^{p-2}\frac{1}{(1+\omega)^\gamma}\left(v_{i-\gamma,k} + v_{i+\gamma,k}\right) +$$

$$+\frac{1}{2(1+\omega)}\left(v_{i-1,k} + v_{i+1,k}\right) \qquad (p = 1, 2, \ldots). \tag{3}$$

This equation is explicit, stable for any $\omega > 0$, and has error $O((\Delta x)^2)$.

Theorem 2. *If the initial function $f(x)$ satisfies the condition*

$$2\int_0^1 f(\xi)\sin\pi\xi\, d\xi > 0,$$

then, for sufficiently large t $(t \geq t_1)$ and sufficiently small Δx, the ordinary explicit equation obtained from (2) with $\alpha = 0$ (see equation (5)) gives, for $\omega < 6$, an approximation to the solution of problem (1) from below, while the explicit equation (3) gives one from above.

The proof of the theorem is based on a comparison of the exact and approximate solutions and on the fact that, for sufficiently large t $(t \gg t_0 > t_1)$ and all x in the interval $(0, 1)$, one has

$$\operatorname{sgn}\frac{\partial^2 u}{\partial t^2} = \operatorname{sgn} 2\int_0^1 f(\xi)\sin\pi\xi\, d\xi\, e^{-\pi^2 t}\sin\pi x.$$

Remarks. 1) The classical implicit equation (4), under the conditions of Theorem 2, gives an approximation to the solution of problem (1) from above.

2) In the general case of a parabolic equation one may assert that the classical explicit and implicit methods have a "tendency" to enclose the solution in a bracket; moreover, the faster the Fourier coefficients in the expansion of the initial function in the corresponding eigenfunctions decrease, the sooner one may expect the bracket to occur.

We give an example illustrating the validity of Theorem 2 (see Table 1; owing to the symmetry of the initial function, only half of the table is reproduced). Here, for each step in t, the first and third rows denote the solutions obtained by the classical explicit method (with $\omega = 2$) and by method (3) with $-\omega = 1!$, respectively; the middle rows give the exact values; $f(x) = 4x(1-x)$, $\Delta x = 1/8$.

As is seen from Table 1, Theorem 2 is valid even for all (and not only for sufficiently large) t. The latter circumstance is explained by the rapid decrease of the Fourier coefficients of the initial function $f(x)$.

Table 1

t	x	0	0.4375	0.7500	0.9375	1.0000
Δt		0	0.3438	0.6250	0.8125	0.8750
Δt		0	0.3470	0.6320	0.8136	0.8750
Δt		0	0.3479	0.6416	0.8186	0.8828
$2\Delta t$		0	0.2891	0.5312	0.6954	0.7500
$2\Delta t$		0	0.2921	0.5371	0.6996	0.7559
$2\Delta t$		0	0.2976	0.5486	0.7114	0.7724
$3\Delta t$		0	0.2461	0.4531	0.5938	0.6406
$3\Delta t$		0	0.2485	0.4592	0.6005	0.6496
$3\Delta t$		0	0.2555	0.4734	0.6166	0.6720
$4\Delta t$		0	0.2100	0.3867	0.5069	0.5468
$4\Delta t$		0	0.2130	0.3939	0.5148	0.5572
$4\Delta t$		0	0.2207	0.4093	0.5344	0.5803

3. As was indicated in the preceding section, the ordinary implicit equation

$$\omega\left(v_{i,k+1}-v_{i,k}\right)=v_{i-1,k+1}-2v_{i,k+1}+v_{i+1,k+1} \tag{4}$$

and the explicit equation

$$\omega\left(v_{i,k+1}-v_{i,k}\right)=v_{i-1,k}-2v_{i,k}+v_{i+1,k} \tag{5}$$

give, for $\Delta t>(\Delta x)^2/6$, i.e. in practically the most interesting cases, approximations to problem (1) from above and below, respectively. In this connection it is natural, for the numerical integration of problem (1), to use equations (4) and (5) alternately:

$$\begin{aligned}\omega\left(v_{i,2k+1}-v_{i,2k}\right)&=v_{i-1,2k+1}-2v_{i,2k+1}+v_{i+1,2k+1},\\ \omega\left(v_{i,2k+2}-v_{i,2k+1}\right)&=v_{i-1,2k+1}-2v_{i,2k+1}+v_{i+1,2k+1},\end{aligned} \tag{6}$$

since in this case, for each pair of steps along the t-axis, a certain compensation of errors will take place.

Theorem 3. *The alternating use of equations (4) and (5) (see (6)), for constant (or changing after an even number of steps)* Δt*, provides a method for solving problem (1) that is stable for every* $\omega>0$.

The proof of the theorem is based on the fact that the eigenvalues of the transition matrix from the $2k$-th layer to the $(2k+2)$-nd, defined by the formula

$$\left(1-\frac{4\Delta t}{(\Delta x)^2}\sin^2\frac{p\pi\Delta x}{2}\right)\Big/\left(1+\frac{4\Delta t}{(\Delta x)^2}\sin^2\frac{p\pi\Delta x}{2}\right)\qquad (p=1,2,\dots,n),$$

for all Δt and Δx, do not exceed 1 in absolute value.

Method (6), being absolutely stable, requires, in comparison with the ordinary implicit method (4), a significantly smaller number of arithmetic operations and, at the same time, because of the mutual compensation of errors noted above, this method is more accurate.

Method (6) generalizes to the m-dimensional case; stability will follow from the fact that

$$\left|1-\frac{4\Delta t}{(\Delta x)^2}\sum_{j=1}^{m}\sin^2\frac{p_j\pi\Delta x}{2}\right|\Big/\left|1+\frac{4\Delta t}{(\Delta x)^2}\sum_{j=1}^{m}\sin^2\frac{p_j\pi\Delta x}{2}\right|<1,$$

$$p_j=1,2,\dots,n;\qquad j=1,2,\dots,m.$$

4. Finally, we indicate the following procedure for applying the ordinary explicit (5) and implicit (4) equations:

$$v_{2i,\,2k+1}=\left(1-\frac{2}{\omega}\right)v_{2i,\,2k}+\frac{1}{\omega}\left(v_{2i-1,\,2k}+v_{2i+1,\,2k}\right),\qquad 1\leqslant i\leqslant\left[\frac{n}{2}\right];$$

$$v_{2i-1,\,2k+1}=\frac{v_{2i-1,\,2k}+\frac{1}{\omega}\left(v_{2i-2,\,2k+1}+v_{2i,\,2k+1}\right)}{1+\frac{2}{\omega}},\qquad 1\leqslant i\leqslant\left[\frac{n+1}{2}\right];$$

$$v_{2i-1,\,2k+2}=\left(1-\frac{2}{\omega}\right)v_{2i-1,\,2k+1}+\frac{1}{\omega}\left(v_{2i-2,\,2k+1}+v_{2i,\,2k+1}\right),\qquad 1\leqslant i\leqslant\left[\frac{n+1}{2}\right];$$

$$v_{2i,\,2k+2}=\frac{v_{2i,\,2k+1}+\frac{1}{\omega}\left(v_{2i-1,\,2k+2}+v_{2i+1,\,2k+2}\right)}{1+\frac{2}{\omega}},\qquad 1\leqslant i\leqslant\left[\frac{n}{2}\right];\qquad (7)$$

$$0\leqslant k\leqslant\left[\frac{T}{2\Delta t}\right]-1.$$

Method (7) (whose stability condition, at least, is $\omega\geqslant 2$) is already explicit. It requires the same number of arithmetic operations as the ordinary explicit method (5), and at the same time, by what was said in Sec. 2, is more accurate.

Received
13 VIII 1957

REFERENCES

[1] V. K. Saul' ev, DAN, 117, No. 1 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.