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Abstract

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A GENERALIZATION OF V. A. MARKOV' S INEQUALITIES

(Presented by Academician S. N. Bernstein on 14 I 1958)

Theorem. If on the interval $[-1,1]$ a polynomial $P_n(x)$ of degree not exceeding n satisfies the inequality

$$|P_n(x)| \leq |\alpha x + i\sqrt{1-x^2}| \quad (\alpha > 0), \tag{1}$$

then

$$|P_n^{(k)}(x)| \leq M_n^{(k)}(1) = \frac{\alpha+1}{2} T_n^{(k)}(1) + \frac{\alpha-1}{2} T_{n-2}^{(k)}(1) \quad (k = 1, \dots, n), \tag{2}$$

where $T_n(x) = \cos n \arccos x$. Equality in (2) is attained only for polynomials $P_n(x) = \gamma M_n(x)$, $|\gamma| = 1$,

$$M_n(x) = \frac{\alpha+1}{2} T_n(x) + \frac{\alpha-1}{2} T_{n-2}(x) \tag{3}$$

at the points $x = \pm 1$.

For $\alpha = 1$ the right-hand side of (1) is equal to unity, and inequalities (2) become V. A. Markov' s inequalities. For $k = 1$ and 2 and any $\alpha > 0$ the theorem was proved in my note (^{1}).

Put

$$M_n(x) = \Re\{(\alpha x + i\sqrt{1-x^2})[T_{n-1}(x) + iS_{n-1}(x)]\},$$

$$L_n(x) = \sqrt{1-x^2} N_{n-1}(x) = \Im\{(\alpha x + i\sqrt{1-x^2})[T_{n-1}(x) + iS_{n-1}(x)]\}, \tag{4}$$

where $S_n(x) = \sin n \arccos x$. The functions $M_n(x)$ and $N_{n-1}(x)$ are polynomials of degrees n and $n-1$, respectively; all their zeros lie in the interval $(-1,1)$

and mutually interlace (see, for example, $(^{2})$). It is not difficult to show that from (4) there follow equality (3) and the equality

$$L_n(x) = \frac{\alpha+1}{2} S_n(x) + \frac{\alpha-1}{2} S_{n-2}(x). \tag{5}$$

If we put

$$H_k(x) = |M_n^{(k)}(x) + iL_n^{(k)}(x)| \quad (k = 1, \dots, n); \tag{6}$$

$$\Phi_k(x) = \begin{cases} H_k(x), & \text{for } \xi_1^{(k)} \leq x \leq \xi_{n-k+1}^{(k)}, \\ |M_n^{(k)}(x)|, & \text{for } -\infty < x \leq \xi_1^{(k)},\ \xi_{n-k+1}^{(k)} \leq x < +\infty, \end{cases} \tag{7}$$

where $\xi_1^{(k)}$ and $\xi_{n-k+1}^{(k)}$ are the extreme zeros of the function $L_n^{(k)}(x)$ lying on the interval $[-1, 1]$, then from my previous results $(^{3-5})$ it follows that, for polynomials satisfying (1), the following estimates of their derivatives are valid:

$$|P_n^{(k)}(x)| \leq \Phi_k(x) \quad (k = 1, \dots, n;\ -\infty < x < \infty). \tag{8}$$

It was originally shown by S. N. Bernstein $(^{6})$ that on the interval $[-1, 1]$ for the first derivative the inequality $|P_n'(x)| \leqslant H_1(x)$ holds.

Our theorem will follow directly from inequality (8), if we show that the even continuous function $\Phi_k(x)$ increases monotonically for $x > 0$. Moreover, since all zeros of $M_n^{(k)}(x)$ lie in the interval $(\xi_1^{(k)}, \xi_{n-k+1}^{(k)})$ $(^{3})$, it is necessary to establish the increase of $\Phi_k(x)$ only on the interval $[0, \xi_{n-k+1}^{(k)}]$.

Let us prove that $H_k^2(x)$ is expanded in the interval $(-1, 1)$ in a Taylor series in even powers of x with positive coefficients. In doing so we shall rely on the fact established in the work of A. Schaeffer and R. Duffin $(^{7})$ that

$$W_{n,k}(x) = [T_n^{(k)}(x)]^2 + [S_n^{(k)}(x)]^2 = \sum_{p=0}^{\infty} a_{p,k} x^{2p}, \qquad a_{p,k} > 0 \tag{9}$$

$$(k = 1, 2, \dots, n;\ p = 0, 1, 2, \dots).$$

Obviously, the function $H_k^2(x)$ can be written in the form

$$\begin{aligned} H_k^2(x) = &\left\{ \frac{\alpha}{2}[T_n^{(k)}(x) + T_{n-2}^{(k)}(x)] + \frac{1}{2}[T_n^{(k)}(x) - T_{n-2}^{(k)}(x)] \right\}^2 \\ &+ \left\{ \frac{\alpha}{2}[S_n^{(k)}(x) + S_{n-2}^{(k)}(x)] + \frac{1}{2}[S_n^{(k)}(x) - S_{n-2}^{(k)}(x)] \right\}^2 . \end{aligned} \tag{10}$$

On the one hand, the identities

$$T_n(x) + T_{n-2}(x) = 2xT_{n-1}(x), \qquad T_n(x) - T_{n-2}(x) = -\frac{2}{n-1}(1-x^2)T'_{n-1}(x),$$

$$S_n(x) + S_{n-2}(x) = 2xS_{n-1}(x), \qquad S_n(x) - S_{n-2}(x) = -\frac{2}{n-1}(1-x^2)S'_{n-1}(x). \tag{11}$$

hold.

On the other hand, the functions $T_n(x)$ and $S_n(x)$ satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2y = 0, \tag{12}$$

therefore $T_n^{(k)}(x)$ and $S_n^{(k)}(x)$ satisfy the equation

$$(1-x^2)y^{(k+2)} - (2k+1)xy^{(k+1)} + (n^2-k^2)y^{(k)} = 0. \tag{13}$$

From the identities (11) we obtain

$$[T_n^{(k)}(x) + T_{n-2}^{(k)}(x)]^2 + [S_n^{(k)}(x) + S_{n-2}^{(k)}(x)]^2 =$$

$$= 4\left[x^2W_{n-1,k}(x) + kx\frac{d}{dx}W_{n-1,k-1}(x) + k^2W_{n-1,k-1}(x)\right]. \tag{14}$$

From (13) we obtain

$$[(1-x^2)T'_{n-1}(x)]^{(k)} = -[xT_{n-1}^{(k)}(x) + (n^2-2n+k)T_{n-1}^{(k-1)}(x)],$$

$$[(1-x^2)S'_{n-1}(x)]^{(k)} = -[xS_{n-1}^{(k)}(x) + (n^2-2n+k)S_{n-1}^{(k-1)}(x)]. \tag{15}$$

Applying (11) and (15), we may write

$$[T_n^{(k)}(x) - T_{n-2}^{(k)}(x)]^2 + [S_n^{(k)}(x) - S_{n-2}^{(k)}(x)]^2 =$$

$$= \frac{4}{(n-1)^2}\left[x^2W_{n-1,k}(x) + (n^2-2n+k)x\frac{d}{dx}W_{n-1,k-1}(x) + \right.$$

$$+(n^2-2n+k)^2 W_{n-1,k-1}(x)\Big] \, ; \tag{16}$$

$$\{[T_n^{(k)}(x)]^2-[T_{n-2}^{(k)}(x)]^2\}+\{[S_n^{(k)}(x)]^2-[S_{n-2}^{(k)}(x)]^2\}=$$

$$=\frac{2}{n-1}\Big[2x^2 W_{n-1,k}(x)+(n^2-2n+2k)x\frac{d}{dx}W_{n-1,k-1}(x)+$$

$$+\,2k(n^2-2n+k)W_{n-1,k-1}(x)\Big] \, . \tag{17}$$

From (14), (16), and (17), taking (9) into account, we conclude that for $k=1,\dots,n-2$ the function $H_k^2(x)$ expands on $(-1,1)$ into an even power series with positive coefficients, and hence for $1\le k\le n-2$ the theorem is proved. For $k=n$ the theorem follows directly from (8), since at $x=1$ we have $|P_n^{(n)}(1)|\le M_n^{(n)}(1)$. For $k=n-1$ the result again follows easily from (8), for the maximum of the linear function $P_n^{(n-1)}(x)$ on the interval $[-1,1]$ is attained at one of the endpoints of the interval; but at the points $x=\pm1$ we have $|P_n^{(n-1)}(\pm1)|\le M_n^{(n-1)}(1)$. In the proof of the theorem it was implicitly assumed that $n\ge 2$, but it is clear that for $n=1$ and 2 inequalities (2) are simple consequences of inequalities (8).

I note that, in the question of a constructive characterization of functions given on a finite interval by means of their approximations by polynomials, investigated recently by V. K. Dzyadyk [8] and A. F. Timan [9], an important role is played by estimates of the successive derivatives of polynomials satisfying the inequalities

$$|P_n(x)|\le\left|\frac{x}{n}+i\sqrt{1-x^2}\right|^{\rho}, \qquad \rho>0, \qquad -1\le x\le 1.$$

In the works mentioned, neither an extremal polynomial for this problem, for any values of ρ, nor exact upper bounds for the derivatives $P_n^{(k)}(x)$ on the interval $[-1,1]$ were given.

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Note: Figure translations are in progress. See original paper for figures.

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