MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. I. PEROV

ON A TWO-POINT BOUNDARY-VALUE PROBLEM

(Presented by Academician P. S. Aleksandrov on 9 VI 1958)

The problem of solutions of the differential equation

$$y'' = f(x, y, y'), \tag{1}$$

satisfying the boundary conditions

$$y(a) = y(b) = 0 \quad (a < b). \tag{2}$$

is investigated.

The question of solutions of equation (1) passing through two prescribed points is, in an obvious way, reduced to the indicated problem.

The boundary-value problem (1)–(2) was studied already by S. N. Bernstein ([1]), who indicated the first conditions for existence and uniqueness of a solution of this problem. Among subsequent investigations we note the works ([2]–[8]). In these works other existence and uniqueness theorems were given. Hammerstein ([9]) and M. A. Krasnosel' skii ([5]) indicated some conditions under which problem (1)–(2) has a nonunique solution.

Below new existence theorems, uniqueness theorems, and nonuniqueness theorems are proposed. For some cases, when the solution is nonunique, lower estimates of the number of solutions are found and some properties of these solutions are studied.

1. In what follows, D denotes the domain $a \leq x \leq b$, $-\infty < y, y' < +\infty$. Throughout the article it is assumed that $f(x, y, y')$ is continuous in D.

By $\alpha_i(x, y, y')$ $(i = 1, 2)$ we denote functions continuous in D, satisfying the Lipschitz condition

$$|\alpha_i(x, y_1, y_1') - \alpha_i(x, y_2, y_2')| \leq L(|y_1 - y_2| + |y_1' - y_2'|)$$

and possessing the property of positive homogeneity:

$$\alpha_i(x, ty, ty') \equiv t\alpha_i(x, y, y') \quad (t \geq 0).$$

Theorem 1. *Suppose $f(x, y, y')$ satisfies the inequalities*

$$f(x, y, y') \geq \alpha_1(x, y, y') + \beta_1(x) \quad \text{for } y \geq 0,$$

$$f(x, y, y') \leq \alpha_2(x, y, y') + \beta_2(x) \quad \text{for } y \leq 0,$$

where $\alpha_i(x, y, y')$ are defined above, and $\beta_i(x)$ are summable on $[a, b]$. Suppose the equation

$$u'' = \alpha_i(x, u, u')$$

for $i = 1$ has a positive solution on $[a, b]$, and for $i = 2$ has a negative solution on $[a, b]$. Then the boundary-value problem (1)–(2) has at least one solution.

It can be shown that, for a special choice of the functions $\alpha_i(x, y, y')$, assertions close to those obtained in ([3], [5], [7], [8]) follow from this theorem.

2. A function $a(x)$ continuous on $[a, b]$ will be called **regular** if the equation

$$y'' = a(x)y \tag{3}$$

has no nonzero solutions satisfying the boundary conditions (2). We shall say that a regular function $a(x)$ has **index** k if the solution of equation (3) satisfying the initial condition

$$y(a) = 0, \qquad y'(a) = 1, \tag{4}$$

has k zeros on the interval (a, b).

The regular functions form, in the space of functions continuous on $[a, b]$, an open set G, decomposing into a countable number of connected components G_k. Each component G_k consists of all regular functions of one and the same index k ($k = 0, 1, 2, ...$); the component G_0 is convex, the others are nonconvex.

From the classical Sturm theorems it follows that the component G_0, together with each function $a_0(x)$, contains all such functions $a(x)$ that

$$a_0(x) \leqslant a(x) \qquad (a \leqslant x \leqslant b). \tag{5}$$

Hence it follows that $a(x) \in G_0$ if

$$-\frac{\pi^2}{(b-a)^2} < a(x). \tag{6}$$

The components G_k $(k = 1, 2, ...)$ have the property that from the inequalities

$$a_k(x) \leqslant a(x) \leqslant a_k^*(x) \qquad (a \leqslant x \leqslant b), \tag{7}$$

where $a_k(x)$, $a_k^*(x) \in G_k$, it follows that $a(x) \in G_k$. Therefore $a(x) \in G_k$ if

$$-\frac{(k+1)^2\pi^2}{(b-a)^2} < a(x) < -\frac{k^2\pi^2}{(b-a)^2}. \tag{8}$$

Let us emphasize that not every function in G_k satisfies inequality (8).

Theorem 2. Suppose $f(x, y, y')$ satisfies in D the inequality

$$\operatorname{sign} y \cdot f(x, y, y') \geqslant a_0(x)|y| + \beta_0(x), \tag{9}$$

where $a_0(x) \in G_0$, $\beta_0(x)$ is summable on $[a, b]$, or the inequalities

$$a_k(x)|y| + \beta(x) \leqslant \operatorname{sign} y \cdot f(x, y, y') \leqslant a_k^*(x)|y| + \beta^*(x), \tag{10}$$

where $a_k(x)$ and $a_k^*(x)$ belong to one and the same component G_k, and $\beta(x)$, $\beta^*(x)$ are summable on $[a, b]$.

Then the boundary-value problem (1)–(2) has at least one solution.

The first assertion of the theorem (when condition (9) is satisfied) follows from Theorem 1. The case when condition (10) is satisfied requires a special proof.

If $f(x, y, y') \equiv f(x, y)$, then condition (9), in which $\beta_0(x)$ is bounded, can be written in the simpler form

$$a_0(x) \leqslant \frac{f(x, y)}{y} \qquad (a_0(x) \in G_0,\ |y| \geqslant M_0), \tag{11}$$

and condition (10), in which $\beta(x)$, $\beta^*(x)$ are bounded, in the form

$$a_k(x) \leqslant \frac{f(x, y)}{y} \leqslant a_k^*(x) \qquad (a_k(x), a_k^*(x) \in G_k;\ |y| \geqslant M_0), \tag{12}$$

where M_0 is some constant.

In particular, for the existence of a solution of the boundary-value problem

$$y'' = f(x,y), \qquad y(a) = y(b) = 0 \tag{13}$$

it is sufficient that $f(x,y)$ satisfy one of the inequalities

$$-\frac{\pi^2}{(b-a)^2} + \varepsilon \leqslant \frac{f(x,y)}{y} \qquad (|y| \geqslant M_0), \tag{14}$$

$$-\frac{(k+1)^2\pi^2}{(b-a)^2} + \varepsilon \leqslant \frac{f(x,y)}{y} \leqslant -\frac{k^2\pi^2}{(b-a)^2} - \varepsilon \qquad (|y| \geqslant M_0), \tag{15}$$

where ε is some positive number.

3. The conditions stated above for the existence of a solution of the boundary-value problem (1)–(2) are, in essence, restrictions on the behavior of $f(x,y,y')$ only "at infinity." This is especially clear when considering the boundary-value problem (13) under conditions (11) or (12). Therefore, in Theorems 1 and 2 the solution is, as a rule, not unique. We give a uniqueness condition for the boundary-value problem

$$y'' = p(x)y' + f(x,y), \qquad y(a) = y(b) = 0. \tag{16}$$

Theorem 3. *Let $p(x)$ and $f(x,y)$ be continuously differentiable in D and satisfy one of the inequalities*

$$a_0(x) \leqslant f'_y(x,y) - \frac{1}{2}p'(x) + \frac{1}{4}p^2(x), \tag{17}$$

where $a_0(x) \in G_0$, or

$$a_k(x) \leqslant f'_y(x,y) - \frac{1}{2}p'(x) + \frac{1}{4}p^2(x) \leqslant a_k^*(x), \tag{18}$$

where $a_k(x), a_k^*(x) \in G_k$.
Then the boundary-value problem (16) has at most one solution.

Let $p(x) \equiv 0$. Then the existence of a solution of the boundary-value problem (13) under the conditions of Theorem 3 follows from Theorem 2. Thus, for the existence and uniqueness of a solution of the boundary-value problem (13) it is sufficient that one of the inequalities

$$a_0(x) \leqslant f'_y(x,y) \qquad (a_0(x) \in G_0), \tag{19}$$

$$a_k(x) \leqslant f'_y(x,y) \leqslant a_k^*(x) \qquad (a_k(x), a_k^*(x) \in G_k) \tag{20}$$

be fulfilled.

For the case of continuous $f'_y(x,y)$, condition (19) contains Picard' s condition ([10]) (see ([4])) for the existence and uniqueness of a solution of problem (13) (Picard assumed that $f'_y(x,y)$ is bounded and positive).

4. Suppose now that one solution of problem (1)–(2) is known. Without loss of generality one may assume that the known solution is the zero solution, i.e. $f(x,0,0) \equiv 0$. Here we shall indicate conditions for the existence of nonzero solutions of problem (1)–(2).

We shall say that $f(x,y,y')$ **satisfies the 0-condition at infinity** if it satisfies the conditions of Theorem 1 (in particular, satisfies condition (9) of Theorem 2). We shall say that $f(x,y,y')$ **satisfies the k-condition at infinity** ($k = 1, 2, \ldots$) if condition (10) is fulfilled.

We shall say that $f(x,y,y')$ **satisfies the l-condition at zero** if $f(x,0,0) \equiv 0$, if $f(x,y,y')$ is continuously differentiable with respect to y, y' in the domain $a \leqslant x \leqslant b,\ |y| + |y'| < \varepsilon$, and, finally, if the solution of the equation

$$u'' = f'_y(x,0,0)u + f'_{y'}(x,0,0)u', \tag{21}$$

satisfying the initial condition $u(a) = 0,\ u'(a) = 1$, has exactly l zeros in the interval (a,b), and $u(b) \neq 0$.

Theorem 4. Suppose the function $f(x,y,y')$ satisfies the k-condition at infinity, and at zero satisfies the l-condition. Suppose $k \neq l$.

Then problem (1)–(2) has solutions different from the zero solution; the number of nonzero solutions is not less than $2|k-l|$; for each $j = 1, \ldots, |k-l|$ there correspond at least two solutions having exactly $m - j$ zeros in the interval (a,b), where $m = \max\{k,l\}$.

Suppose, for example, that condition (9) is fulfilled. Let

$$f(x,0,0) \equiv f'_{y'}(x,0,0) = 0, \quad f'_y(x,0,0) \equiv \alpha$$

(α is constant). Then it follows from Theorem 4 that, when

$$-\frac{(l+1)^2\pi^2}{(b-a)^2} < \alpha < -\frac{l^2\pi^2}{(b-a)^2}, \tag{22}$$

the boundary-value problem (1)–(2) has at least $2l$ distinct nonzero solutions. Under similar conditions this problem was considered by M. A. Krasnosel' skii [5], from whose results it follows only that nonzero solutions exist if condition (22) is satisfied, in which l is odd; moreover, the method applied in [5] does not give an estimate of the number of nonzero solutions.

The author takes this opportunity to express sincere gratitude to M. A. Krasnosel' skii for the advice and help given in writing this note.

Voronezh State
University

Received
10 V 1958

CITED LITERATURE

[1] S. N. Bernstein, *Uspekhi Mat. Nauk*, **8**, 32 (1941).
[2] G. D. Birkhoff, O. D. Kellog, *Trans. Am. Math. Soc.*, **23**, 96 (1922).
[3] L. Tonelli, *Ann. Sc. Norm. Super. Pisa*, (2), **8**, 75 (1939).
[4] J. Sansone, *Ordinary Differential Equations*, Vol. 2, IL, 1954.
[5] M. A. Krasnosel' skii, *Izv. AN SSSR, Ser. Mat.*, **20**, 241 (1956).
[6] Z. F. Surikova, *On a Boundary-Value Problem for Ordinary Differential Equations*, dissertation, Moscow, 1954.
[7] Chan Chun, *DAN*, **113**, No. 6, 1227 (1957).
[8] H. Epheser, *Math. Zs.*, **61**, No. 4 (1955).
[9] A. Hammerstein, *Jahresber. d. deutsch. Math. Ver.*, **39**, 59 (1930).
[10] É. Picard, *J. Math. Pure et Appl.*, (4), **9**, 217 (1893).

Note: Figure translations are in progress. See original paper for figures.