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G. E. KUZMAK

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Abstract

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G. E. KUZMAK

ASYMPTOTIC SOLUTIONS OF THE EQUATION OF MOTION OF A DISSIPATIVE SYSTEM WITH ONE DEGREE OF FREEDOM AND SLOWLY VARYING PARAMETERS

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In the present paper we consider the equation

$$\frac{d^2y}{dt^2} + f\left(\tau, \frac{dy}{dt}\right) + \varepsilon F(\tau, y) = 0; \tag{1}$$

ε is a small parameter; $\tau = \varepsilon t$ is slow time. This equation may be interpreted as the equation of motion of a material point of mass equal to unity under the action of a principal force $-f(\tau, dy/dt)$, depending on the velocity dy/dt (a dissipative force), and a small force $-\varepsilon F(\tau, y)$. The purpose of the work is to compute expressions that uniformly approximate the solution of equation (1) and its derivative with an error of order ε on a time interval of order $1/\varepsilon$.

The problem is considered under the following assumptions:

a) $f(\tau, 0) = 0$.

b) $f_z(\tau, 0) \geq \Delta > 0$ (the subscript z denotes derivatives of the function $f(\tau, dy/dt)$ with respect to dy/dt).

c) The function $f(\tau, dy/dt)$ is considered for $0 \leq |dy/dt| \leq w_t$, $0 \leq \tau \leq \tau_0$. In this domain it does not vanish, except for the value $dy/dt = 0$, and is analytic in dy/dt together with its first and second derivatives with respect to τ.

d) The function $F(\tau, y)$ is defined for $0 \leq |y| \leq w$ and $0 \leq \tau \leq \tau_0$, is analytic in this domain with respect to y, and is continuous in τ.

In items c) and d), w_t and w are certain constants.

To solve the problem posed, the method of "model equations" ([1],[2]) is used. As the "model" equation we choose the equation

$$\varphi^2(\tau)\frac{\partial^2 y_0}{\partial \omega^2} + f\left[\tau, \varphi(\tau)\frac{\partial y_0}{\partial \omega}\right] = 0. \tag{2}$$

The function $\varphi(\tau)$ is chosen by means of the equality

$$\varphi(\tau) = f_z(\tau, 0). \tag{3}$$

Then the solution of equation (2) is written in the form

$$y_0(\tau, \omega) = B_0(\tau) + A[\tau, e^{-\omega - c(\tau)}]. \tag{4}$$

Here the functions $B_0(\tau)$ and $c(\tau)$ are arbitrary, while the function $A[\tau, e^{-\omega-c(\tau)}]$ is determined when solving equation (2) and may be expanded in a series in powers of $e^{-\omega-c(\tau)}$, convergent for $0 < \Omega \leq \omega < \infty$ and $0 \leq \tau \leq \tau_0$ (Ω is a constant independent of ε):

$$A[\tau, e^{-\omega-c(\tau)}] = \sum_{n=1}^{\infty} B_n(\tau) e^{-n[\omega + c(\tau)]}. \tag{5}$$

From the general theorems on the dependence of solutions of differential equations on a parameter and from equation (2) it follows that, if in the expression for $y_0(\tau, \omega)$ one expresses τ and ω in terms of t by means of the relations

$$\frac{d\tau}{dt} = \varepsilon, \qquad \frac{d\omega}{dt} = \varphi(\tau), \tag{6}$$

then the function of t thus obtained will differ from the solution of equation (1) by quantities of order ε on each time interval of order unity. In order that this function of t approximate the solution of equation (1) uniformly on a time interval of order $1/\varepsilon$, it is necessary to determine the functions $B_0(\tau)$ and $c(\tau)$ in the appropriate way. To derive the conditions for determining the functions $B_0(\tau)$ and $c(\tau)$, substitute the function $y_0(\tau, \omega)$, where τ and ω are expressed in terms of t by means of (6), into equation (1). We obtain:

$$\varphi^2(\tau) \frac{\partial^2 y_0}{\partial \omega^2} + f\left[\tau, \varphi(\tau) \frac{\partial y_0}{\partial \omega}\right] + \varepsilon \Phi(\tau, \omega) + O(\varepsilon^2) = 0.$$

Here

$$\Phi(\tau, \omega) = 2\varphi(\tau) \frac{\partial^2 y_0}{\partial \omega \, \partial \tau} + \varphi'(\tau) \frac{\partial y_0}{\partial \omega} + f_z\left[\tau, \varphi(\tau) \frac{\partial y_0}{\partial \omega}\right] \frac{\partial y_0}{\partial \tau} + F(\tau, y_0). \tag{7}$$

We obtain the necessary conditions by considering the terms of order ε. Under the assumptions made above, it follows from equalities (4) and (5) that the function $\Phi(\tau, \omega)$ can be represented in the form of a series in powers of $e^{-\omega}$, convergent for $0 < \Omega \leq \omega < \infty$ and $0 \leq \tau \leq \tau_0$:

$$\Phi(\tau,\omega) = \sum_{n=0}^{\infty} \Phi_n(\tau) e^{-n\omega}. \tag{8}$$

The coefficients $\Phi_n(\tau)$ are expressed in terms of the functions $B_0(\tau)$ and $c(\tau)$, and also in terms of the known functions entering equation (1). From condition b) and equalities (3) and (6) it follows that, as t increases, each subsequent term of the series (8) decays faster than the preceding one. Accordingly, as conditions for determining the functions $B_0(\tau)$ and $c(\tau)$, it is expedient to take the conditions that the coefficients $\Phi_0(\tau)$ and $\Phi_1(\tau)$ be equal to zero, since the corresponding terms of the series (8) decay more slowly than the other terms. The resulting relations for the functions $B_0(\tau)$ and $c(\tau)$ have the form

$$\varphi(\tau) B_0'(\tau) + F[\tau, B_0(\tau)] = 0,$$

$$c(\tau) = \int_0^{\tau} \frac{\varphi'(\tau) - F_y[\tau, B_0(\tau)] - f_{zz}(\tau,0) F[\tau, B_0(\tau)]}{\varphi(\tau)}\, d\tau + \ln\left|\frac{B_1(\tau)}{B_1(0)}\right|. \tag{9}$$

Primes here denote derivatives with respect to τ.

We note that, under condition (9), the inequality

$$|\Phi(\tau,\omega)| \le M_\Phi e^{-2\omega}; \tag{10}$$

holds, where M_Φ is a constant independent of ε.

Theorem. *If the functions $f(\tau, dy/dt)$ and $F(\tau, y)$ satisfy the conditions stated above, and the arbitrary functions $B_0(\tau)$ and $c(\tau)$ entering the solution (4) of the "reference" equation are determined by means of equalities (9), then for $|\varepsilon| \le \varepsilon_0$ the functions*

$$y_0(t) = y_0(\tau,\omega), \qquad \left(\frac{dy}{dt}\right)_0 = \varphi(\tau)\frac{\partial y_0}{\partial \omega},$$

where $\tau = \varepsilon t$, $\omega = \int \varphi(\tau)\, dt$, approximate, respectively, the solution of equation (1) and its derivative with an error of order ε on a time interval of order $1/\varepsilon$.

For the proof, represent the function $y(t)$ and its derivative* in the form

$$y(t) = y_0(t) + \varepsilon Y(t), \qquad \frac{dy}{dt} = \left(\frac{dy}{dt}\right)_0 + \varepsilon\left[\frac{\partial y_0}{\partial \tau} + \frac{dY}{dt}\right]. \tag{11}$$

Substituting equalities (11) into equation (1), we obtain an equation for the function $Y(t)$:

$$\frac{d^2Y}{dt^2} + f_z\left[\tau, \varphi(\tau)\frac{\partial y_0}{\partial \omega}\right]\frac{dY}{dt} = -\Phi(t) + \varepsilon\Psi\left(t, Y, \frac{dY}{dt}, \varepsilon\right). \tag{12}$$

Here the function $\Psi(t, Y, dY/dt, \varepsilon)$ is expressed in terms of the derivatives of the functions $f(\tau, dy/dt)$ and $F(\tau, y)$, in which y and dy/dt are replaced by means of (11), and also in terms of the function $y_0(\tau, \omega)$ and its derivatives; the function $\Phi(t)$ is the function (7), in which τ and ω are expressed through t by means of equalities (6).

Next replace equation (12) by an equivalent system of integral equations under the conditions $Y(0) = 0,\ dY/dt|_{t=0} = 0$:

$$Y = \int_0^t Q_0(t,\xi)\left[-\Phi(\xi) + \varepsilon\Psi\left(\xi, Y, \frac{dY}{dt}, \varepsilon\right)\right]d\xi,$$

$$\frac{dY}{dt} = \int_0^t Q_1(t,\xi)\left[-\Phi(\xi) + \varepsilon\Psi\left(\xi, Y, \frac{dY}{dt}, \varepsilon\right)\right]d\xi. \tag{13}$$

Here

$$v(t) = \int_\infty^t \exp\left[-\int_0^t f_z\left[\tau, \varphi(\tau)\frac{\partial y_0}{\partial \omega}\right]dt\right],$$

$$Q_0(t,\xi) = \frac{v(t) - v(\xi)}{dv(\xi)/d\xi}, \qquad Q_1(t,\xi) = \frac{dv(t)/dt}{dv(\xi)/d\xi}.$$

It is not difficult to show that, in the region of variation of the arguments under consideration $0 \leqslant \xi \leqslant t \leqslant \tau_0/\varepsilon$, the kernels $Q_0(t,\xi)$ and $Q_1(t,\xi)$ are bounded by some constant M_Q independent of ε.

Let H denote a positive number greater than $M_\Phi M_Q/2\Delta$. If the values $|Y|$ and $|dY/dt|$ do not exceed H, then, under the conditions of the theorem, for $0 \leqslant t \leqslant \tau_0/\varepsilon$ and $|\varepsilon| \leqslant \varepsilon_0$ the function $\Psi(t, Y, dY/dt, \varepsilon)$, first, is bounded by a constant M_Ψ, and, second, satisfies the Lipschitz condition in Y and dY/dt with constant L. We note that the constants H, M_Ψ, and L can be chosen independent of ε.

Apply to equations (13) the method of successive approximations:

$$Y^{(0)} = -\int_0^t Q_0(t,\xi)\Phi(\xi)d\xi, \qquad \frac{dY^{(0)}}{dt} = -\int_0^t Q_1(t,\xi)\Phi(\xi)d\xi,$$

$$Y^{(n+1)} = Y^{(0)} + \varepsilon \int_0^t Q_0(t,\xi)\Psi\left(\xi, Y^{(n)}, \frac{dY^{(n)}}{dt}, \varepsilon\right) d\xi, \tag{14}$$

$$\frac{dY^{(n+1)}}{dt} = \frac{dY^{(0)}}{dt} + \varepsilon \int_0^t Q_1(t,\xi)\Psi\left(\xi, Y^{(n)}, \frac{dY^{(n)}}{dt}, \varepsilon\right) d\xi \qquad (n = 0, 1, 2, ...).$$

Using inequality (10) for $\Phi(t)$, the boundedness of the kernels, and the above-indicated properties of the function $\Psi(t, Y, dY/dt, \varepsilon)$, from (14) we obtain the following estimates:

$$|Y^{(0)}| \leqslant \frac{M_\Phi M_Q}{2\Delta}, \qquad |Y^{(n+1)} - Y^{(n)}| \leqslant \frac{M_\Psi}{2L} \frac{(\varepsilon M_Q 2Lt)^{n+1}}{(n+1)!} \qquad (n = 0, 1, 2, ...). \tag{15}$$

Exactly the same estimates are obtained for the derivatives. These inequalities are valid under the condition that none of the approximations exceeds H. Below we shall define the number $\tau_* \leqslant \tau_0$ so that for $0 \leqslant t \leqslant \tau_*/\varepsilon$ this condition will be satisfied.

In order to estimate $Y(t)$, represent it in the form

$$Y(t) = Y^{(0)} + \sum_{n=0}^{\infty} \left(Y^{(n+1)} - Y^{(n)}\right). \tag{16}$$

Using inequalities (15), for $0 \leqslant t \leqslant \tau_*/\varepsilon$ and $|\varepsilon| \leqslant \varepsilon_0$, we obtain

$$|Y(t)| \leqslant \frac{M_\Phi M_Q}{2\Delta} + \frac{M_\Psi}{2L}\left(e^{2M_Q L\tau} - 1\right). \tag{17}$$

We now define the number τ_*. From (16) we have that $|Y^{(n+1)}(t)|$, as well as $|Y(t)|$, satisfies inequality (17). Introduce the number τ_1 by means of the equality

$$\frac{M_\Phi M_Q}{2\Delta} + \frac{M_\Psi}{2L}\left(e^{2M_Q L\tau_1} - 1\right) = H.$$

Note that $\tau_1 > 0$ because the number H is greater than $M_\Phi M_Q/2\Delta$. If, as the number τ_*, we choose the smaller of the numbers τ_0 and τ_1, then it is evident that $|Y^{(n+1)}(t)|$ and $|dY^{(n+1)}/dt|$ will not exceed H for $0 \leqslant t \leqslant \tau_*/\varepsilon$ and $|\varepsilon| \leqslant \varepsilon_0$. Accordingly, on this same time interval we have

$$|Y(t)| \leqslant H, \qquad \left|\frac{dY}{dt}\right| \leqslant H.$$

These inequalities, by virtue of (11) and the boundedness of the function $\partial y_0/\partial\tau$, which holds under the conditions of the theorem, prove the required assertion.

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References

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2. G. E. Kuzmak, *Prikl. Mat. Mekh.*, **21**, no. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.

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