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Abstract

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ON EXPANSION IN EIGEN VECTOR-FUNCTIONS OF NON-SELF-ADJOINT DIFFERENTIAL SYSTEMS OF SECOND ORDER

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The question of expanding functions on the half-axis in solutions of the non-self-adjoint problem for the differential equation

$$y'' - p(x)y + \lambda^2 y = 0 \tag{1}$$

under the condition $hy(0)+y'(0) = 0$ was studied in the works of M. A. Naimark ([1,2]). Subsequently B. Ya. Levin ([3]), by another method—namely, using transformation operators—constructed an analogue of the Fourier inversion formulas with the aid of special solutions of the equation and obtained expansion theorems.

In the present note, the results of B. Ya. Levin are extended to the case of a system of differential equations with complex coefficients. Instead of a system it is more convenient to consider the matrix differential equation

$$Y'' - YP(x) + \lambda^2 Y = 0; \tag{2}$$

$P(x)$ is a square matrix of order m with complex components.

We shall call a solution of equation (2) satisfying the condition*

$$\lim_{x\to\infty} \left|Y(x,\lambda) - e^{i\lambda x} I\right| = 0$$

a **principal solution**.

The basic fact for all subsequent arguments is

Theorem 1**. *If the condition*

$$\int_0^\infty x\,|P(x)|\,dx < \infty \tag{3}$$

is satisfied, then there exists a principal solution of equation (2), which is represented in the form

$$Y(x,\lambda) = e^{i\lambda x} I + \int_x^\infty K(x,t) e^{i\lambda t}\, dt \quad (\operatorname{Im}\lambda \geqslant 0), \tag{4}$$

where the kernel-matrix $K(x,t)$ *is continuous in the domain* $0 \leq x \leq t < \infty$ *and*

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* By the absolute value of a matrix $A = (a_{ij})_{i,j=1}^n$ we mean

$$A = \left(\sum_{i,j}^m |a_{ij}|^2 \right)^{1/2}.$$

** Theorem 1 was proved for the scalar case by B. Ya. Levin ([3]) under a somewhat greater restriction on $P(x)$. In the form given here it was stated by Z. S. Agranovich and V. A. Marchenko.

satisfies the conditions

$$\int_0^\infty \int_x^\infty |K(x,t)|^2\, dt\, dx < \infty, \qquad \int_x^\infty |K(x,t)|\, dt < \infty, \qquad x \geq 0.$$

Replacing λ by $-\lambda$, we obtain the second solution $Y(x,-\lambda)$ $(\operatorname{Im}\lambda \geq 0)$ of equation (2), linearly independent of the principal one. The resolvent $R(s,x)$ of the kernel $K(s,x)$ of the integral equation*

$$\mathbf{f}(x) + \int_0^x K(s,x)\mathbf{f}(s)\, ds = \mathbf{g}(x)$$

is a continuous function for $0 < s \leq x < \infty$, and

$$\int_0^\infty \int_0^x |R(s,x)|^2\, ds\, dx < \infty.$$

Denote by $L^2_m(0,\infty)$ the space of m-dimensional vector-functions, each of whose components belongs to L^2 on the positive ray and is equal to zero on the negative ray; by $L^{2(+)}_m$, the m-dimensional vector space in which the components of the vector are holomorphic and bounded in the upper half-plane, and on the real axis belong to L^2.

Theorem 2. The transformation

$$\vec{\varphi}(\lambda) = \frac{1}{\sqrt{2\pi}} \,\mathrm{l.\,i.\,m.} \int_0^{\infty} Y(x,\lambda)\mathbf{f}(x)\,dx, \tag{5}$$

in which $Y(x,\lambda)$ is the principal solution, maps the space $L_m^{2(+)}$ into $L_m^2(0,\infty)$, and the transformation

$$\mathbf{f}(x) = \frac{1}{\sqrt{2\pi}} \,\mathrm{l.\,i.\,m.} \int_{-\infty}^{\infty} Z(x,\lambda)\vec{\varphi}(\lambda)\,d\lambda, \tag{6}$$

in which

$$Z(x,\lambda) = e^{-i\lambda x} I + \int_0^x R(s,x) e^{-i\lambda s}\,ds,$$

maps $L_m^{2(+)}$ into $L_m^2(0,\infty)$. Moreover,** $c_1\|\vec{\varphi}\| \leq \|\mathbf{f}\| \leq c_2\|\vec{\varphi}\|$, where c_1 and c_2 are positive constants.

Relation (6) gives not only the inversion formula for (5), but also the expansion of $\mathbf{f}(x)$ in the matrices $Z(x,\lambda)$. The matrices $Z(x,\lambda)$, generally speaking, are not solutions of equation (2), but are related to the equation in the following way:

Theorem 3. The matrix-function

$$Z(x,\lambda) = e^{-i\lambda x} I + \int_0^x R(s,x) e^{-i\lambda s}\,ds,$$

which gives the inversion formula (6), satisfies the nonhomogeneous equation

$$Z'' - P(x)Z + \lambda^2 Z = -i\lambda R(0,x) + R_s'(0,x).$$

* $\mathbf{f}(x)$ and $\mathbf{g}(x)$ are vector-functions.

** The norm of a vector-function is defined as follows: $\|\mathbf{f}\|^2 = \int_{-\infty}^{\infty} |\mathbf{f}(x)|^2\,dx$.

The expansion in the fundamental solutions $Y(x,\lambda)$ is given by the theorem:

Theorem 4. Every vector-function $\mathbf{f}(x) \in L_m^2(0,\infty)$ admits an expansion in the fundamental solutions of equation (2), namely:

$$\mathbf{f}(x) = \frac{1}{\sqrt{2\pi}} \,\mathrm{l.i.m.} \int_{-\infty}^{\infty} Y(x,\lambda)\,\vec{\psi}(\lambda)\,d\lambda, \tag{7}$$

where

$$\vec{\psi}(\lambda) = \frac{1}{\sqrt{2\pi}} \,\text{l.i.m.} \int_0^\infty Z(x,\lambda)\,\mathbf{f}(x)\,dx. \tag{8}$$

Under the stronger restriction than (3),

$$\int_0^\infty (1+x)|P(x)|\,dx < \infty \tag{9}$$

a linear combination of the independent solutions $Y(x,\lambda)$ and $Y(x,-\lambda)$ makes it possible to obtain a solution satisfying a boundary condition at zero that does not depend on λ. Denote by $\Phi(x,\lambda)$ the solution of equation (2) with initial conditions $\Phi(0,\lambda) = I,\ \Phi'(0,\lambda) = -H$; the solution $\Phi(x,\lambda)$ has the form

$$\Phi(x,\lambda) = \zeta(\lambda)\big[Y_H^{-1}(\lambda)Y(x,\lambda) + Y_H^{-1}(-\lambda)Y(x,-\lambda)\big],$$

where

$$Y_H(\lambda) = -\frac{1}{i\lambda}\big[Y(0,\lambda)H + Y'(0,\lambda)\big],$$

$$\zeta(\lambda) = \big[Y_H^{-1}(\lambda)Y(0,\lambda) + Y_H^{-1}(-\lambda)Y(0,-\lambda)\big]^{-1}.$$

The complex roots of the equation $\det Y_H(\lambda) = 0$ constitute the entire discrete spectrum of the operator $\mathcal{L}$, defined in the space $L_m^2(0,\infty)$ by the differential expression $ly = y'' - \widetilde{P}(x)y$ and the boundary condition $\widetilde{H}y(0) + y'(0) = 0$. Here the sign $\sim$ denotes passage to the transposed matrix. If $\det Y_H(\lambda) \neq 0$ on the real axis, then the discrete spectrum consists of a finite number of points. Denote by G_m^2 the orthogonal complement in $L_m^2(0,\infty)$ to the subspace spanned by the vectors, complex conjugate to the vectors, constituting invariant subspaces of the operator $\mathcal{L}$. Further, let H_m^2 be the vector space of even functions belonging to L_m^2 on the real axis. An essential role in what follows is played by the solution $V(x,\lambda)$ of the inhomogeneous matrix differential equation

$$V'' - P(x)V + \lambda^2 V = A(x)$$

with initial conditions $V(0,\lambda) = I,\ V'(0,\lambda) = -H$. The matrix $A(x)$ is uniquely determined, according to a certain rule, by the matrices $P(x)$ and H.

Theorem 5. If on the real axis $\det Y_H(\lambda) \neq 0$, then the transformation

$$\mathbf{F}(\lambda) = \frac{1}{\sqrt{2\pi}} \,\text{l.i.m.} \int_0^\infty \Phi(x,\lambda)\mathbf{f}(x)\,dx$$

maps G_m^2 into H_m^2, and the formula

$$\mathbf{f}(x) = \sqrt{\frac{2}{\pi}}\,\text{l.i.m.} \int_0^\infty V(x,\lambda)\,d\sigma(\lambda)\cdot\mathbf{F}(\lambda) \qquad (\sigma'(\lambda) = \xi^{-1}(\lambda)) \tag{10}$$

gives the inverse transformation from H_m^2 into G_m^2.

Theorem 6. If on the real axis $\det Y_H(\lambda) \neq 0$, then for any $\mathbf{f}(x) \in G_n^2$ the expansion

$$\mathbf{f}(x) = \sqrt{\frac{2}{\pi}}\,\text{l. i. m.} \int_0^\infty d\sigma(\lambda)\,\Phi(x,\lambda)\mathbf{q}(\lambda), \tag{11}$$

holds, where

$$\mathbf{q}(\lambda) = \frac{1}{\sqrt{2\pi}}\,\text{l. i. m.} \int_0^\infty V(x,\lambda)\mathbf{f}(x)\,dx.$$

In the scalar case the functions $\Phi(x,\lambda)$ and $V(x,\lambda)$ coincide. For a system the following theorems hold:

Theorem 7. If $P(x)$ is permutable with H, then the matrix-function $V(x,\lambda)$ satisfies the homogeneous equation

$$V'' - P(x)V + \lambda^2 V = 0.$$

The kernel $K(x,t)$ of Theorem 1 defines the Volterra operator

$$W(\mathbf{f}) = \mathbf{f}(x) + \int_x^\infty K(x,t)\bar{\mathbf{f}}(t)\,dt,$$

which transforms $L_n^p(0,\infty)$ $(1 \le p < \infty)$ into itself and carries every solution $B\sin(\lambda x + \alpha)$ of the matrix equation $Y'' + \lambda^2 Y = 0$ (B is a constant matrix) into a solution $\Psi(x,\lambda)$ of equation (2), satisfying the condition

$$\lim_{x\to\infty} |\Psi(x,\lambda) - B\sin(\lambda x + \alpha)| = 0.$$

Application of the operator W makes it possible to prove the following theorem on equiconvergence.

Theorem 9. If $\det Y_H(\lambda) \neq 0$ for $\operatorname{Im}\lambda = 0$, then the expansions (10) and (11) of the vector-function $\mathbf{f}(x)$ converge in the same sense (uniformly or in L_n^p) as the representation of this function by the Fourier integral

$$\mathbf{f}(x) = \int_0^\infty \cos \lambda x \cdot \vec{\eta}(\lambda)\, d\lambda.$$

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References

1. M. A. Naimark, DAN, 89, 213 (1953).
2. M. A. Naimark, Tr. Moskovsk. matem. obshch., 3, 181 (1954).
3. B. Ya. Levin, DAN, 106, 187 (1956).

* All theorems concerning the transformation by means of the solution $\Phi(x, \lambda)$ extend to the case of the boundary condition $\Phi(0, \lambda) = 0$.

Note: Figure translations are in progress. See original paper for figures.