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# MATHEMATICAL PHYSICS

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**Abstract**

**Full Text**

## MATHEMATICAL PHYSICS

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### ON THE CONNECTION BETWEEN THE S-MATRIX AND THE POTENTIAL FOR A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR

(Presented by Academician V. I. Smirnov, 32 III 1958)

We shall consider the one-dimensional Schrödinger equation

$$Ly = -\frac{d^2}{dx^2}y + q(x)y = k^2y \quad (1)$$

on the whole axis  $-\infty < x < \infty$ , and require that the condition

$$\int_{-\infty}^{\infty} (1 + |x|) |q(x)| dx < \infty. \quad (2)$$

be satisfied.

1. Denote by  $f_1(x, k)$  the solution for which  $\lim_{x \rightarrow \infty} e^{-ikx} f_1(x, k) = 1$ , and by  $f_2(x, k)$  the solution for which  $\lim_{x \rightarrow -\infty} e^{ikx} f_2(x, k) = 1$ . Under condition (2), these solutions exist for all  $k$  in the upper half-plane  $\text{Im } k \geq 0$ . B. Ya. Levin <sup>(1)</sup> proved that the representation

$$f_1(x, k) = e^{ikx} + \int_x^{\infty} A_1(x, y) e^{iky} dy, \quad (3)$$

holds, where

$$\int_a^{\infty} dx \int_x^{\infty} dy |A_1(x, y)|^2 \leq C_a, \quad a > -\infty. \quad (4)$$

In an analogous way one can show that

$$f_2(x, k) = e^{-ikx} + \int_{-\infty}^x A_2(x, y) e^{-iky} dy; \quad (5)$$

$$\int_{-\infty}^b dx \int_{-\infty}^x dy |A_2(x, y)|^2 \leq C_b, \quad b < \infty. \quad (6)$$

For real  $k \neq 0$ , the solutions  $f_1(x, k)$  and  $f_1(x, -k) = \overline{f_1(x, k)}$ , or  $f_2(x, k)$  and  $f_2(x, -k) = \overline{f_2(x, k)}$ , form a complete system, so that every solution can be represented as their linear combination. In particular:

$$\begin{aligned} f_1(x, k) &= c_{11}(k)f_1(x, k) + c_{12}(k)f_1(x, -k); \\ f_1(x, k) &= c_{22}(k)f_2(x, k) + c_{12}(k)f_2(x, -k). \end{aligned} \quad (7)$$

**Lemma 1.** The coefficients  $c_{ij}(k)$  are continuous functions of  $k$  on the whole axis  $-\infty < k < \infty$ , except for  $k = 0$ , and  $kc_{ij}(k)$  are continuous up to  $k = 0$ . The following relations hold:

$$\begin{aligned} c_{ij}(k) &= \overline{c_{ij}(-k)}, \quad i, j = 1, 2; \quad c_{12}(k) = c_{21}(k); \\ c_{11}(k) &= -c_{22}(-k); \quad |c_{12}|^2 = 1 + |c_{11}|^2 = 1 + |c_{22}|^2. \end{aligned} \quad (8)$$

For large  $|k|$ ,

$$c_{ii}(k) = O\left(\frac{1}{|k|}\right), \quad i = 1, 2; \quad c_{12}(k) = 1 + O\left(\frac{1}{|k|}\right). \quad (9)$$

It follows from (8) that  $c_{12}(k) \neq 0$ . Denote

$$s_{21}(k) = s_{12}(k) = \frac{1}{c_{12}(k)}, \quad s_{ii}(k) = \frac{c_{ii}(k)}{c_{12}(k)}, \quad i = 1, 2. \quad (10)$$

Then from (7) we have:

$$\begin{aligned} s_{12}(k)f_2(x, k) &= s_{11}(k)f_1(x, k) + f_1(x, -k); \\ s_{12}(k)f_1(x, k) &= s_{22}(k)f_2(x, k) + f_2(x, -k). \end{aligned} \quad (11)$$

We see that the solution  $s_{12}(k)f_1(x, k)$ , which as  $x \rightarrow \infty$  behaves as  $s_{12}(k)e^{ikx}$ , as  $x \rightarrow -\infty$  behaves as  $s_{22}(k)e^{-ikx} + e^{ikx}$ ; precisely this type of solution is considered in quantum scattering theory on a one-dimensional potential barrier (2). The functions  $s_{11}(k)$  and  $s_{22}(k)$  may be called the amplitudes of the reflection coefficient to the right and to the left, and the function  $s_{12}(k)$  the amplitude of the transmission coefficient. Conditions (8) are equivalent to the

assertion that the symmetric matrix  $S(k)$  with entries  $s_{ij}(k)$  is unitary, with  $S(-k) = S^*(k) = S^{-1}(k)$ . The matrix  $S(k)$  is called the scattering operator, or the  $S$ -matrix, for the operator  $L$ . The unitarity condition is equivalent to the fulfillment of the following relations:

$$|s_{12}|^2 = 1 - |s_{11}|^2 = 1 - |s_{22}|^2; \quad s_{12}(-k)s_{11}(k) + s_{12}(k)s_{22}(-k) = 0. \quad (12)$$

**Lemma 2.** For all real  $k$ , except possibly  $k = 0$ ,

$$||s_{ii}(k)|| < 1, \quad i = 1, 2. \quad (13)$$

If  $|s_{ii}(0)| = 1$ , then

$$s_{11}(0) = s_{22}(0) = -1. \quad (14)$$

The function  $s_{12}(k)$  is the boundary value of a function that is analytic in the upper half-plane  $\text{Im } k \geq 0$ , except possibly for a finite number of points on the imaginary axis  $k = i\tau_l$ ,  $l = 1, \dots, n$ , where it has simple poles. For large  $|k|$ ,

$$s_{ii}(k) = O\left(\frac{1}{|k|}\right), \quad i = 1, 2, \text{Im } k = 0; \quad s_{12}(k) = 1 + O\left(\frac{1}{|k|}\right), \quad \text{Im } k \geq 0. \quad (15)$$

The points  $k_l$  correspond to those values of  $k$  for which the solutions  $f_1(x, k)$  and  $f_2(x, k)$  are linearly dependent,

$$f_1(x, i\tau_l) = \alpha_l f_2(x, i\tau_l), \quad l = 1, \dots, n. \quad (16)$$

In this case

$$\text{Res } s_{12}(k)|_{k=i\tau_l} = i\gamma_l = i \left[ \int_{-\infty}^{\infty} f_1(x, i\tau_l) f_2(x, i\tau_l) dx \right]^{-1}. \quad (17)$$

The quantities  $\lambda_l = -\tau_l^2$  are the eigenvalues of the operator  $L$ . We introduce the Fourier transforms of the functions  $s_{ij}(k)$ :

$$F_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{11}(k) e^{ikt} dk; \quad F_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{22}(k) e^{-ikt} dk;$$

$$\Gamma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (s_{12}(k) - 1) e^{-ikt} dk; \quad (18)$$

$F_1(t)$ ,  $F_2(t)$ , and  $\Gamma(t)$  are square-integrable functions of  $t$  on the whole axis, and

$$\Gamma(t) = \sum_{l=1}^n \gamma_l e^{\tau_l t}, \quad t < 0. \quad (19)$$

On the basis of the convolution theorem, relations (11) are equivalent to the following relations:

$$\begin{aligned} \int_{-\infty}^{\infty} A_2(x, t) \Gamma(t - y) dt + A_2(x, y) + \Gamma(x - y) &= \\ &= \int_{-\infty}^{\infty} A_1(x, t) F_1(t + y) dt + A_1(x, y) + F_1(x + y); \\ \int_{-\infty}^{\infty} A_1(x, t) \Gamma(y - t) dt + A_1(x, y) + \Gamma(y - x) &= \\ &= \int_{-\infty}^{\infty} A_2(x, t) F_2(t + y) dt + A_2(x, y) + F_2(x + y). \end{aligned} \quad (20)$$

Hence, with the aid of (18), (16), and (17), we can obtain the following equations connecting the functions  $A_1(x, y)$ ,  $A_2(x, y)$ , and  $F_1(t)$ ,  $F_2(t)$ :

$$A_1(x, y) + \Omega_1(x + y) + \int_x^{\infty} A_1(x, t) \Omega_1(t + y) dt = 0, \quad x < y; \quad (21)$$

$$A_2(x, y) + \Omega_2(x + y) + \int_{-\infty}^x A_2(x, t) \Omega_2(t + y) dt = 0, \quad x > y. \quad (22)$$

Here

$$\Omega_1(t) = F_1(t) + \sum_{l=1}^n m_{l_1} e^{-\tau_l t}; \quad \Omega_2(t) = F_2(t) + \sum_{l=1}^n m_{l_2} e^{\tau_l t}, \quad (23)$$

where

$$m_{l_1} = \left( \int_{-\infty}^{\infty} [f_1(x, i\tau_l)]^2 dx \right)^{-1}; \quad m_{l_2} = \left( \int_{-\infty}^{\infty} [f_2(x, i\tau_l)]^2 dx \right)^{-1}. \quad (24)$$

From relations (16) and (17) it follows that

$$m_{l_1} m_{l_2} = \gamma_l^2, \quad l = 1, \dots, n. \quad (25)$$

Equations of the type (21) were used for solving the inverse problem for the radial equation by V. A. Marchenko <sup>(3)</sup> and were studied in detail in the works of Z. S. Agranovich and V. A. Marchenko <sup>(4)</sup>. On the basis of their investigation we may assert that the functions  $F_1(t)$  and  $F_2(t)$  are once differentiable and

$$\int_a^\infty (1 + |t|)|F_1'(t)| dt < C_a, \quad a > -\infty;$$

$$\int_{-\infty}^b (1 + |t|)|F_2'(t)| dt < C_b, \quad b < \infty. \quad (26)$$

2. The relations obtained make it possible to solve the problem of reconstructing the potential from the  $S$ -matrix. This problem was studied by Kay and Moses <sup>(5)</sup>, who obtained an equation of the type (22) and, under the assumption of solvability of this equation, investigated what properties of  $s_{22}(k)$  guarantee that the corresponding potential will be identically zero to the left of a certain point. Our investigation has a different aim, namely, to determine what properties the elements of the  $S$ -matrix must possess in order that the corresponding potential satisfy only condition (2). The inverse problem for equation (1) in another formulation was studied by A. Sh. Blokh <sup>(6)</sup>.

Let a function  $s_{11}(k)$  be given, satisfying conditions (13) and (15), and let  $2n$  arbitrary positive parameters  $\tau_{l_1}$  and  $m_{l_1}$  be given, with no two of the  $\tau_{l_1}$  equal. If, from these data, one constructs the function  $\Omega_1(t)$  by formula (23), then equation (21), for all  $x$ ,  $-\infty < x < \infty$ , is uniquely solvable. An analogous assertion holds for equation (22), if one starts from a function  $s_{22}(k)$  satisfying conditions (13) and (15), and arbitrary positive  $\tau_{l_2}$  and  $m_{l_2}$ , again with no two of the  $\tau_{l_2}$  equal. The results of Z. S. Agranovich and V. A. Marchenko allow one to assert that the obtained solutions  $A_1(x, y)$  and  $A_2(x, y)$  satisfy, respectively, conditions (4) and (6), and that the functions  $f_1(x, k)$  and  $f_2(x, k)$  constructed by formulas (3) and (5) are solutions of the differential equations with potentials  $q_1(x)$  and  $q_2(x)$ , where

$$\int_a^\infty (1 + |x|)|q_1(x)| dx < C_a, \quad a > -\infty;$$

$$\int_{-\infty}^b (1 + |x|)|q_2(x)| dx < C_b, \quad b < \infty. \quad (27)$$

The solution of the inverse problem is completed on the basis of the following lemma:

**Lemma 3.** Let the functions  $s_{ij}(k)$  possess the properties listed in Lemma 2 and be connected by relations (12). Let, further,  $\tau_{l_1} = \tau_{l_2} = \tau_l$ ,  $l = 1, \dots, n$ , with  $k_l = i\tau_l$  coinciding with the points at which  $s_{12}(k)$  has singularities, and

let the relations (25) hold for  $m_{l_1}$  and  $m_{l_2}$ . Then the relations (11) hold for the functions  $f_1(x, k)$  and  $f_2(x, k)$ .

Our result may be formulated in the form of the following theorem:

**Theorem.** In order that a symmetric unitary matrix  $S(k)$  be the  $S$ -matrix of an operator of type  $L$ , it is necessary and sufficient that its elements possess the properties indicated in Lemma 2 and satisfy condition (26). In this case the potential is determined uniquely if, in addition to the  $S$ -matrix,  $n$  arbitrary positive numbers are given, where  $n$  is the number of discrete eigenvalues.

3. It should be noted that the requirements of Lemma 2 and the unitarity condition make it possible to reconstruct the  $S$ -matrix from one of the elements  $s_{11}(k)$  or  $s_{22}(k)$ , provided the number of singularities of  $s_{12}(k)$  in the upper half-plane is specified. Thus, suppose  $s_{11}(k)$  is given and has properties (13) and (15). Then

$$s_{12}(k) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |s_{11}(k')|^2)}{k' - k} dk' \right] \prod_{l=1}^n \frac{k + i\tau_l}{k - i\tau_l}, \quad \text{Im } k > 0; \quad (28)$$

$$s_{12}(k) = \lim_{\varepsilon \rightarrow 0} s_{12}(k + i\varepsilon), \quad \text{Im } k = 0,$$

has the properties indicated in Lemma 2, and, if we set

$$s_{22}(k) = -s_{11}(-k)s_{12}(k)(s_{12}(-k))^{-1}, \quad (29)$$

then we obtain an admissible  $S$ -matrix, provided the Fourier transforms  $s_{22}(k)$  and  $s_{11}(k)$  satisfy condition (26). In this sense one may say that the potential is reconstructed from the reflection amplitude in one direction.

It is worth noting that if no properties are required of  $s_{11}(k)$  except square integrability and decay for large  $|k|$ , then the corresponding equations are solvable, and the resulting potential is a generalized function of the type of the derivative of a square-integrable function. In particular, one may obtain a potential having a  $\delta$ -type singularity.

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