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Abstract

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MATHEMATICS

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ON THE EXPANSION OF A POSITIVE KERNEL INTO A BILINEAR SERIES

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The subject of the present note is a certain generalization of Mercer' s theorem on the expansion of a positive kernel into a bilinear series.

Let $K(x, s)$ be a continuous, symmetric, and positive kernel, given in the square $a \leq x, s \leq b$. Consider functions of bounded variation $p_i(x)$, given on the interval $\langle a, b\rangle$, and define from them the functions $\Phi_i(x)$ and the numbers a_{ij}:

$$\Phi_i(x) = \int_a^b K(x, s)\, dp_i(s), \qquad a_{ij} = \int_a^b \Phi_i(x)\, dp_j(x).$$

We shall call the sequence of functions $\{p_i(x)\}$ a **sequence of ties** if the determinants

$$\Delta_n = \left|a_{ij}\right|_{i,j=1}^n, \qquad n = 1, 2, ...$$

are nonzero.

It is not difficult to prove that if the kernel is degenerate, then the sequence of ties will consist of a finite number of functions, and also that for any nondegenerate kernel one can always construct an infinite sequence of ties. In what follows we shall consider only infinite sequences of ties, since for degenerate kernels the results we obtain are trivial.

From the kernel $K(x, s)$ and the sequence of ties we construct kernels $R_n(x, s)$, defined by the equalities:

$$R_0(x,s) = K(x,s), \qquad R_n(x,s) = \frac{1}{\Delta_n}\begin{vmatrix} K(x,s) & \Phi_1(x)\ldots\Phi_n(x) \\ \Phi_1(s) & \\ \vdots & \Delta_n \\ \Phi_n(s) & \end{vmatrix}, \qquad n > 0. \tag{1}$$

From Sylvester' s identity, applied to the determinants occurring in (1), there follows the recurrence relation

$$R_n(x,s) = R_{n-1}(x,s) - \frac{\int_a^b R_{n-1}(x,t)\,dp_n(t)\int_a^b R_{n-1}(s,t)\,dp_n(t)}{\int_a^b\int_a^b R_{n-1}(x,s)\,dp_n(x)\,dp_n(s)}, \tag{2}$$

with the aid of which, applying induction, it is easy to prove the positivity of the kernels $R_n(x,s)$. We also note that from equality (1) there follows the relation

$$\int_a^b R_n(x,s)\,dp_i(s) = 0, \qquad i = 1,2,\ldots,n.$$

For an arbitrary sequence of ties the following theorem is valid.

Theorem 1. A symmetric, positive, and continuous kernel $K(x,s)$ is representable by the bilinear series

$$K(x,s) = R(x,s) + \sum_{i=1}^{\infty}\frac{\int_a^b R_{i-1}(x,t)\,dp_i(t)\int_a^b R_{i-1}(s,t)\,dp_i(t)}{\int_a^b\int_a^b R_{i-1}(x,s)\,dp_i(x)\,dp_i(s)}, \tag{3}$$

uniformly convergent in the two variables. Here $R(x,s)$ is also a symmetric, positive, and continuous kernel, and the equalities

$$\int_a^b R(x,s)\,dp_i(s) = 0, \qquad i = 1,2,\ldots \tag{4}$$

hold.

We indicate the main stages of the proof. From the positivity of the kernel and the recurrence relation (2) it follows that

$$0 \le R_n(x,x) \le R_{n-1}(x,x);$$

this leads to the convergence of the sequence $R_n(x,s)$ for $x = s$. From the same recurrence relation one easily obtains inequalities proving the convergence of the sequence of functions under consideration throughout the square. These same inequalities show that the sequence $\{R_n(x,s)\}$ will converge uniformly in the square if it converges uniformly on the diagonal of the square.

Since the functions $R_n(x,x)$ form a monotonically decreasing sequence of continuous functions, by Dini' s theorem, in order to prove uniform convergence it suffices to prove the continuity of the limiting function.

By successive application of relation (2), we represent the kernel $K(x,s)$ as the sum of two positive kernels, one of which is $R_n(x,s)$, and prove that, if a positive continuous kernel is equal to the sum of positive continuous kernels depending on n and having limits, then the limiting kernels of these sequences are also continuous. The assertions just listed show the uniform convergence of the series (3) and the properties of the function $R(x,s)$ stated in the theorem.

From Theorem 1 there follows immediately the theorem on the expansion of a positive kernel in fundamental functions of a loaded integral equation with an arbitrary distribution function.*

Indeed, let $\sigma(t)$ be a function of bounded variation, and let $\{\lambda_i\}$ and $\{\varphi_i(x)\}$ be, respectively, the sets of all characteristic numbers and fundamental functions of the integral equation

$$\varphi(x) = \lambda \int_a^b K(x,t)\varphi(t)\,d\sigma(t), \tag{5}$$

where the functions $\varphi_i(x)$ are chosen so that

$$\int_a^b \varphi_i(t)\varphi_j(t)\,d\sigma(t) = \begin{cases} 0, & i \neq j, \\ \operatorname{sign}\lambda_i, & i = j. \end{cases}$$

If in expression (3) we put

$$p_i(x) = \int_a^x \varphi_i(t)\,d\sigma(t), \tag{6}$$

* This result was first obtained by M. G. Krein ([1]), proceeding from other considerations.

then we directly obtain the required expansion

$$K(x,s) = R(x,s) + \sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(s)}{|\lambda_i|}.$$

It can be proved that the function $R(x, s)$ entering the last expression is such that

$$\int_a^b R^2(x,t)\,d\sigma(t) = 0.$$

Thus, $R(x, s) = 0$, if $\sigma(t)$ is a monotone function and at least one of the numbers x, s belongs to the set of points of increase of the function $\sigma(t)$.

We shall call a sequence of constraints **complete on a set** E, if outside this set $dp_i(x) = 0$ $(i = 1, 2, ...)$ and if every continuous function $f(x)$ satisfying the conditions

$$\int_a^b f(x)\,dp_i(x) = 0 \qquad (i = 1, 2, ...),$$

is identically equal to zero on E.

Let $R(x, s)$ be the residual function corresponding to a sequence of constraints complete on E, and let $\sigma(t)$ be an arbitrary monotone function for which the set of points of increase coincides with E.

We prove that if $R^*(x, s)$ is the residual function entering the expansion of the kernel in the fundamental functions of equation (5), then

$$R^*(x,s) = R(x,s).$$

The equality obtained leads to the theorems:

Theorem 2. *The residual functions corresponding to two different sequences of constraints, complete on one and the same set, coincide.*

Theorem 3. *The residual function entering the expansion of a symmetric, positive, and continuous kernel in the fundamental functions of a loaded integral equation with a monotone distribution function does not depend on the form of the distribution function and is uniquely determined by the set of its points of increase.*

For the approximate solution of certain problems it is essential to indicate such a sequence of constraints that would lead to the best convergence (in the sense defined below) of the series (3).

For this purpose we use the necessary and sufficient conditions indicated in ([2]), which the first n functions of the sequence of constraints must satisfy in order that the maximum of the integral

$$\int_a^b \int_a^b R_n(x,s)\,dQ(x)\,dQ(s) \tag{7}$$

be minimal. Using Courant' s theorem ([3]) on the minimax properties of eigenvalues, it is not difficult to show that the constraints constructed according to rule (6) are the desired ones; however, we have shown that, along with the mentioned constraints, there also exist other constraints solving the posed problem. From the remark made it follows that the expansion of a kernel in the fundamental functions of an integral equation is, in a certain sense, the best, since the maxima of quadratic forms of the form

(7), constructed from the difference of the kernel and the partial sums of the series (3), will be minimal for any n $(n = 1, 2, ...)$.

Let the kernel $K(x, s)$ be the Green' s function of the differential operator L_{2n} under certain Sturm boundary conditions. We shall assume that the sequence of connections is complete on the set E, consisting of a finite number of intervals. If the intervals $\langle \alpha_k, \beta_k \rangle$ supplement the set E to the interval $\langle a, b \rangle$, then the residual function is uniquely determined by the following theorem.

Theorem 4*. a) If x and s belong to the interval $\langle \alpha_k, \beta_k \rangle$, then $R(x, s)$ is the Green' s function of the differential operator L_{2n} on this interval. At the endpoint of the interval under consideration which coincides with the boundary of the interval $\langle \alpha_k, \beta_k \rangle$, $R(x, s)$ satisfies the same boundary conditions as $K(x, s)$; at the endpoint of the interval $\langle \alpha_k, \beta_k \rangle$ which is an interior point of the interval $\langle a, b \rangle$, the function $R(x, s)$ and all its derivatives with respect to x up to order $(n - 1)$ inclusive are equal to zero.

b) $R(x, s) = 0$ if one of the points x, s belongs to the interval $\langle \alpha_k, \beta_k \rangle$ and the other lies outside this interval, and also if one of the points x, s belongs to E.

In conclusion we note that the sufficient conditions, obtained by us in the course of the proof of Theorem 2, for the convergence of the bilinear series to the kernel can be substantially weakened. (In the work of M. G. Krein [5], the necessary and sufficient conditions for such an expansion are indicated in a very general setting.)

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CITED LITERATURE

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* In one special case, an analogous result was obtained by us earlier in connection with the study of the distribution of the eigenvalues of a differential operator with a small coefficient at the highest derivative (4).

Note: Figure translations are in progress. See original paper for figures.