GENERALIZED SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Abstract

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MATHEMATICS

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GENERALIZED SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACE

(Presented by Academician S. L. Sobolev on 5 VI 1958)

Consider the problem

$$x' + A(t)x = f(t, x) \qquad (0 \le t \le T), \tag{1}$$

$$x(0) = x_0 \tag{2}$$

in a Hilbert space H, to which various boundary-value problems for partial differential equations of parabolic type are reduced.

In the present article we study generalized solutions of problem (1)–(2). Apparently, generalized solutions of boundary-value problems from the standpoint of functional analysis were first studied by S. L. Sobolev, whose works served as the starting point for many investigations.

1. We shall say that $x(t)$ is a generalized solution of problem (1)–(2) on $[0, T]$ if $x(t)$ satisfies equation (1) for almost all t, $x(t) =$

$$= x_0 + \int_0^t x' \, dt$$

and x' and $A(t)x$ belong to $B_2([0, T], H)$ (see (1)). This definition of a generalized solution is equivalent to one of the definitions given by O. A. Ladyzhenskaya (2).

Consider the linear equation

$$x' + A(t)x = f(t). \tag{3}$$

Theorem 1. *Let $A(t) = A_1(t) + A_2(t)$ and let the positive-definite self-adjoint operators $A_1(t)$ $(0 \le t \le T)$ have a common domain of definition D. Let $B(t, 0) = A_1(t)A_1^{-1}(0)$, as an operator-function of t, have discontinuities only* *of the first kind (property (a)). Let $D(A_2(t)) \supset D$, and let the function $A_2(t)x$, for every $x \in D$, be strongly measurable and satisfy the inequality*

$$\|A_2(t)x\| \leq \delta\|A_1(t)x\| + C\|x\| \qquad (0 \leq \delta < 1,\ C \geq 0). \tag{4}$$

Then, for arbitrary $f(t) \in B_2([0,T],H)$ and $x_0 \in D(A_1^{1/2}(0))$, there exists a unique generalized solution $x(t)$ of problem (3)–(2). The function $A_1^{1/2}(0)x(t)$ is continuous *.

The proof of this theorem is based on the properties of operators forming an acute angle (^{3}). Theorem 1 is a strengthening of one theorem from (^{2}), where it is assumed that $B(t,0)$ is strongly differentiable and that in inequality (4) the constant $\delta < 1/4$.

2. Below we shall assume that $A(t)$ satisfies the conditions of Theorem 1 and that $A_2(t) = 0$. Denote by $U(t,s)x_0$ the solution of the homogeneous equation satisfying the initial condition $U(s,s)x_0 = x_0$.

Theorem 2. For any $x \in H$, $\alpha \in [0,1/2]$, $0 \leq s \leq t \leq T$, the inequalities

$$\left\|A^{\alpha}(0)[U(t,s) - I]A^{-1/2}(0)x\right\| \leq |t-s|^{1/2-\alpha}K_\alpha\|x\|; \tag{5}$$

$$\left[\int_s^t \left\|A^{1/2+\alpha}(0)U(\xi,s)A^{-1/2}(0)x\right\|^{1/\alpha} d\xi\right]^{\alpha} \leq K_\alpha\|x\|. \tag{6}$$

The properties of the operator $U(t,s)$, under the assumption of strong continuous differentiability of $B(t,0)$, were previously studied by other methods in works ($^{4-6}$). In these works the solution $x(t)$ of the nonhomogeneous equation (3) was constructed in the form

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds = U(t,0)x_0 + Qf(t). \tag{7}$$

Theorem 3. The generalized solution of problem (3)–(2) is representable by formula (7).

Theorem 4. For any $f(t) \in B_2([0,T],H)$, $\alpha \in [0,1/2]$, $0 \leq t \leq t+\Delta t \leq T$, the inequalities

$$\left\|A^{\alpha}(0)[Qf(t+\Delta t) - Qf(t)]\right\| \leq \Delta t^{1/2-\alpha}K_\alpha\left[\int_0^{t+\Delta t}\|f(s)\|^2\,ds\right]^{1/2}; \tag{8}$$

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* The continuity of the function $A_1^{1/2}(0)x(t)$ is a necessary condition for the existence of a generalized solution of problem (1)–(2), if, for example, $A(t)$ satisfies the conditions of Theorem 1.

$$\left[\int_0^t \left\|A^{1/2+\alpha}(0)Qf(\xi)\right\|^{1/\alpha} d\xi\right]^{\alpha} \leq K_\alpha \left[\int_0^t \|f(s)\|^2 \, ds\right]^{1/2}. \tag{9}$$

The nonlinear equation (1) was first studied by M. A. Krasnosel' skii and S. G. Krein, and then by M. A. Krasnosel' skii, S. G. Krein, and the author (^{5,6}).

In order that $x(t)$ be a generalized solution of problem (1)–(2), it is necessary and sufficient that $x(t)$ be a solution of the equation

$$x(t) = U(t,0)x_0 + Qf(t, x(t)) \tag{10}$$

and that $f(t, x(t)) \in B_2([0,T], H)$ (Theorem 3). With the aid of Schauder' s principle and estimates (5), (9), it is proved:

Theorem 5. Suppose that $A^{-1}(0)$ is completely continuous as an operator in H, and that $t, A^{-\alpha}(0)y(t)$ maps $C([0,T],H)$ (see (1)) into $B_2([0,T],H)$, is continuous and bounded for some $\alpha \in [0, 1/2)$.

Then there exists at least one generalized solution of problem (1)–(2), defined on some segment $[0, h]$.

3. We shall say that $A(t)$ has property (b) if $B(t,0)$ has bounded variation; property (c), if $B(t,0)$ satisfies the Lipschitz condition.

Theorem 6. The inequality

$$\left\|A^{\beta}(t)U(t,s)A^{-\alpha}(s)\right\| \leq \frac{K_{\alpha\beta}}{|t-s|^{\beta-\alpha}} \qquad (t > s) \tag{11}$$

is valid for all $0 \leq \alpha \leq \beta \leq 1$ in the case when $A(t)$ has property (b), and for all $0 \leq \alpha \leq 1$, $0 \leq \beta < 1 + \varepsilon$, $\alpha \leq \beta$ in the case when $A(t)$ has property (c).

In the first case the proof is based on the representation of $U(t,s)$ in the form of a multiplicative integral and is carried out by the method developed

in (6). Here the following inequality is used:

$$\|A^{\rho}(t)A^{-\rho}(s)\| \leq \|A(t)A^{-1}(s)\|^{\rho} \quad (0 \leq \rho \leq 1), \tag{12}$$

which follows from Heinz' s inequality (7). We note that the representation of $U(t,s)$ by a product integral holds in the case when $A(t)$ has only property (a).

In the second case, the proof of (11) is based on the study of the operator integral equation

$$U(t,s) = e^{-(t-s)A(\xi)} + \int_s^t e^{-(t-\eta)A(\xi)}[A(\xi) - A(\eta)]U(\eta, s)\, d\eta. \tag{13}$$

From (11) it follows that

$$\|A^{-\alpha}(t)U(t,s)A^{\beta}(t)\| \le \frac{K_{\alpha\beta}}{|t-s|^{\beta-\alpha}}, \tag{14}$$

and also the inequalities

$$\|A^{\alpha}(0)[U(t,s)-I]A^{-\beta}(0)\| \le K_{\alpha\beta}|t-s|^{\beta-\alpha} \quad (0 \le \alpha \le \beta \le 1); \tag{15}$$

$$\|A^{\alpha}(0)[Qf(t+\Delta t)-Qf(t)]\| \le \Delta t^{\gamma-\alpha}K_{\alpha\beta\gamma}\left[\int_0^{t+\Delta t}\|f(s)\|^{\beta}\,ds\right]^{1/\beta} \tag{16}$$

$$\left(\alpha \in [0,1],\ \beta > \frac{1}{1-\alpha},\ \gamma \in \left[\alpha, \frac{\beta-1}{\beta}\right]\right),$$

which strengthen, respectively, (5) and (8).

With the aid of (15) and (16) one proves:

Theorem 7. Let $A(t)$ have property (b) or (c), and let $A^{-1}(0)$ be completely continuous as an operator in H. Suppose that, for some $\alpha \in [0,1]$, $f(t, A^{-\alpha}(0)y(t))$, acting from $C([0,T],H)$ into $B_{\beta}([0,T],H)$ $\left(\beta > \frac{1}{1-\alpha}\right)$, is continuous and bounded*. Finally, suppose that $x_0 \in D(A^{\gamma}(0))$ $(\gamma \ge \alpha)$.

Then there exists at least one solution $x(t)$ of equation (10), defined on some segment $[0,h]$, and such that $f(t,x(t)) \in B_{\beta}([0,h],H)$. If $\gamma, \beta \ge 1/2$, then $x(t)$ is a generalized solution of problem (1)–(2).

4. Let $A(t)$ have property (c). With the aid of Theorem 6 it is proved that, for $t > s$, the operator $U(t,s)$ is continuously differentiable both with respect to t and with respect to s, and satisfies the equations

$$\frac{\partial U(t,s)}{\partial t} + A(t)U(t,s) = 0, \qquad \frac{\partial U(t,s)}{\partial s} - \overline{U(t,s)A(s)} = 0. \tag{17}$$

This fact makes it possible to strengthen the theorems on the existence of classical solutions of linear and nonlinear equations given in (5, 6).

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* In applications to partial differential equations this means that the nonlinear part may contain derivatives with respect to the spatial variables whose order is lower than the order of the differential operator $A(0)$.

Note: Figure translations are in progress. See original paper for figures.

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