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Abstract

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ON SOLUTIONS OF CERTAIN PERTURBED SYSTEMS IN A NEIGHBORHOOD OF PERIODIC MOTIONS

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§ 1. **Statement of the problem.** Consider the unperturbed system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1}$$

and the corresponding perturbed system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{f}(\mathbf{x}, \varepsilon), \tag{2}$$

where $\mathbf{F}, \mathbf{f}, \mathbf{x}$ are n-dimensional vectors, and ε is a small parameter. Suppose that, in some domain, (1) has only periodic solutions:

$$\mathbf{x} = \mathbf{x}_0(\mathbf{c}, \psi), \tag{3}$$

where $\mathbf{c} = \{c_1, c_2, \dots, c_{n-1}\}$; $\psi = \omega(\mathbf{c})t + h$ is the phase of the oscillations; c_i $(i = 1, 2, \dots, n-1)$ and h are n arbitrary constants; the function $\mathbf{x}_0$ is periodic in ψ with period 2π; $\omega(\mathbf{c}) \equiv 2\pi/T(\mathbf{c}) \geq \text{const} > 0$ is the frequency; $T(\mathbf{c})$ is the period of the oscillations. Suppose that the solutions (3) correspond to a complete system of independent integrals of (1) of the form

$$\vec{\Phi}(\mathbf{x}_0) = \mathbf{c}, \qquad \Theta(\mathbf{x}_0) = \psi, \tag{4}$$

where $\vec{\Phi} = \{\Phi_1, \Phi_2, \dots, \Phi_{n-1}\}$, and the integrals $\vec{\Phi}$, $\Theta - \omega t$ correspond, respectively, to the constants $\mathbf{c}, h$ entering into (3). The solutions of (2) (nonperiodic, generally speaking) are considered in a neighborhood of the periodic solutions (3) of system (1). We shall seek an approximate representation of the solutions of the perturbed system (2) in the form (3), where $\mathbf{c} = \mathbf{c}(t, \varepsilon)$, $\psi = \psi(t, \varepsilon)$ are certain unknown functions. Since the functions (3) correspond to the integrals (4), $\mathbf{c}(t, \varepsilon)$, $\psi(t, \varepsilon)$ are obtained if, in $\vec{\Phi}$, Θ, one substitutes, instead of the solutions of (1), the solutions of (2); in general, $\mathbf{c}$ will then no longer be constant, and ψ will differ from $\omega t + h$. Thus the problem consists in finding asymptotic approximations for the functions $\mathbf{c}(t,\varepsilon) = \vec{\Phi}[\mathbf{x}(t,\varepsilon)]$, $\psi(t,\varepsilon) = \Theta[\mathbf{x}(t,\varepsilon)]$ with respect to the small parameter ε, these approximations being constructed for large time intervals $t \sim 1/\varepsilon$. In accordance with what has been said above, this is equivalent to an approximate representation of the solutions (2) in the form (3) by means of variation of the constants $\mathbf{c}$ and of the phase ψ.

§ 2. **Main results.** Suppose that all the functions under consideration are continuous, bounded, and sufficiently smooth, and that all systems of equations satisfy the conditions of existence, uniqueness, and continuous dependence of solutions on initial values and parameters in the corresponding domains. Assume that there exists an open domain G of the space $\{x_1, x_2, \dots, x_n\}$, from all points of whose closure $\overline{G}$ there issue closed trajectories of (1), corresponding to the periodic solutions (3) with integra-

by the integrals (4). Let G_0 be some open subdomain of G, lying inside G together with its boundary. Let $\varepsilon \neq 0$ and be sufficiently small. Introduce the interval $\Delta = [t_1, t_1 + \frac{a}{\varepsilon}]$, where $a > 0$ is an arbitrarily large fixed number, and t_1 is an arbitrary instant of time. Take the integral curves (2), whose trajectories begin at some instant $t_0 \in \Delta$ inside G_0, and consider them either on the whole interval Δ, if for $t \in \Delta$ the trajectory does not leave G_0, or on some part $\Delta_1 = [t_0 - b, t_0 + d] \subset \Delta$ $(b, d \geq 0)$ such that for $t \in \Delta_1$ the trajectory lies in G_0. For these solutions of the perturbed system (2), on the indicated intervals of time, we shall study the asymptotics of $c(t,\varepsilon)$, $\psi(t,\varepsilon)$. In (2) make the change of variables according to formulas (3), where $c = c(t,\varepsilon)$, $\psi = \psi(t,\varepsilon)$ are the functions of interest to us. As a result of this substitution, after some transformations we derive the equations:

$$\dot{c} = (\varepsilon f \vec{\nabla}_x)\vec{\Phi}, \qquad \dot{\psi} = \omega(c) + (\varepsilon f \vec{\nabla}_x)\Theta \quad \left(\vec{\nabla}_x = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right\}\right). \tag{5}$$

The right-hand sides of (5) are expressed, by means of (3), through c, ψ and are periodic in ψ with period 2π. (5) belongs to the systems with a rotating phase studied in (1). Discarding terms of order higher than ε in the equations $c = (\varepsilon f \vec{\nabla}_x)\vec{\Phi}$ and averaging with respect to ψ, we obtain, according to (1), the equations of the zero approximation for c, which are then reduced to the form

$$\dot{c} = \frac{1}{T(c)} \int_0^{T(c)} (\varepsilon f_0 \vec{\nabla}_x)\vec{\Phi}\, dt \qquad (f_0 = f|_{\varepsilon=0}). \tag{6}$$

Thus, equations (6) have been derived, describing the change of $c(t,\varepsilon)$ over large time intervals $t \sim 1/\varepsilon$ with an error $\sim \varepsilon$. The quadratures in (6) depend on c as on a parameter and are taken along the cycles (3) of system (1). The solutions (3) and the integrals (4) of the unperturbed system (1) are regarded as known, and c in (6) changes slowly (since $\dot{c} \sim \varepsilon$); therefore it is simpler to determine the approximations c from (6) than to integrate the perturbed system (2). The equations of higher approximations for c and approximations for ψ (they are not written out here, since in the general case they are rather cumbersome) are also derived from (5) by the averaging method ([1]).

§ 3. **Some special cases.**

1. **Periodic solutions.** Suppose that some solution x_0 of system (1) generates a periodic solution of (2), which tends to x_0 as $\varepsilon \to 0$. From (6) there follows a necessary condition for the existence of such solutions: the values $c = \vec{\Phi}(x_0)$, determining the generating solution x_0, must satisfy the equations

$$A(c) \equiv \int_0^{T(c)} (f_0 \vec{\nabla}_x)\vec{\Phi}\, dt = 0.$$

Periodic solutions of (2) and of more general systems were studied in ([2]), where criteria for the stability of these solutions were also formulated. According to ([2]), a sufficient condition for the existence of periodic solutions of (2) is

$$\frac{D(A_1, A_2, \dots, A_{n-1})}{D(c_1, c_2, \dots, c_{n-1})} \neq 0$$

for $A(c) = 0$.

2. **Canonical systems.** Suppose that the unperturbed system (1) is canonical:

$$\dot{q} = \vec{\nabla}_p H(p, q), \qquad \dot{p} = -\vec{\nabla}_q H(p, q), \tag{7}$$

where q, p are n-dimensional vectors; $H(p, q)$ is the Hamiltonian function; $\vec{\nabla}_q = \left\{\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n}\right\}$; $\vec{\nabla}_p = \left\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \dots, \frac{\partial}{\partial p_n}\right\}$. We write the perturbed system in the form

$$\dot{q} = \vec{\nabla}_p H(p, q) - \varepsilon f_p(p, q, \varepsilon), \qquad \dot{p} = -\vec{\nabla}_q H(p, q) + \varepsilon f_q(p, q, \varepsilon), \tag{8}$$

where $\mathbf{f}_p = \{f_p^{(1)}, f_p^{(2)}, \dots, f_p^{(n)}\}$, $\mathbf{f}_q = \{f_q^{(1)}, f_q^{(2)}, \dots, f_q^{(n)}\}$. If the periodic solutions (7) depend on $2n$ arbitrary constants, then formula (6) is applicable to (8). System (7) has the energy integral $E = H(\mathbf{p}, \mathbf{q})$; therefore from (6), for (8), the energy equation is derived:

$$\dot{E} = \frac{1}{T} \oint (\varepsilon \mathbf{f}_{q_0} d\mathbf{q} + \varepsilon \mathbf{f}_{p_0} d\mathbf{p}) \quad (\mathbf{f}_{p_0} \equiv \mathbf{f}_p|_{\varepsilon=0}, \quad \mathbf{f}_{q_0} \equiv \mathbf{f}_q|_{\varepsilon=0}), \tag{9}$$

where the integral is taken along the cycles (7) in the direction of the unperturbed motion. In item 4 of § 3 the energy equation will be derived for more general systems, of which (8) is a special case, and a physical interpretation of this equation will be given.

3. **Equations of oscillations with slowly varying parameters.** Consider the unperturbed system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \vec{\mu}) \quad (\vec{\mu} = \text{const}), \tag{10}$$

where $\mathbf{x}, \mathbf{F}$ are n-dimensional vectors; $\vec{\mu} = \{\mu_1, \mu_2, \dots, \mu_m\}$ is a set of m parameters. Suppose that (10) has periodic solutions depending on n arbitrary constants and $\vec{\mu}$: $\mathbf{x} = \mathbf{x}_0(\mathbf{c}, \vec{\mu}, \psi)$ ($\mathbf{c} = \{c_1, c_2, \dots, c_{n-1}\}$, $\psi \equiv \omega(\mathbf{c}, \vec{\mu})t + h$), to which there corresponds a complete system of independent integrals $\vec{\Phi}(\mathbf{x}_0, \vec{\mu}) = \mathbf{c}$, $\Theta(\mathbf{x}_0, \vec{\mu}) = \psi$. Consider a perturbed system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \vec{\mu}) + \varepsilon \mathbf{f}(\mathbf{x}, \vec{\mu}, \varepsilon), \qquad \dot{\vec{\mu}} = \varepsilon \vec{\varphi}(\mathbf{x}, \vec{\mu}, \varepsilon), \tag{11}$$

where $\mathbf{f} = \{f_1, f_2, \dots, f_n\}$, $\vec{\varphi} = \{\varphi_1, \varphi_2, \dots, \varphi_m\}$, in which $\vec{\mu}$ are no longer constant, but vary slowly with the small velocity $\varepsilon\vec{\varphi}$. It is required to find approximations for the solutions $\mathbf{x}(t, \varepsilon)$, $\vec{\mu}(t, \varepsilon)$ of system (11), which reduces to constructing approximations for $\vec{\mu}(t, \varepsilon)$, $\mathbf{c}(t, \varepsilon) \equiv \vec{\Phi}[\mathbf{x}(t, \varepsilon), \vec{\mu}(t, \varepsilon)]$, $\psi(t, \varepsilon) \equiv \Theta[\mathbf{x}(t, \varepsilon), \vec{\mu}(t, \varepsilon)]$. This is a special case of the problem of §§ 1 and 2, since one may formally append to (10) the equations $\dot{\vec{\mu}} = 0$ and regard their solutions $\vec{\mu} = \text{const}$ as periodic, and add to the integrals $\vec{\Phi} = \mathbf{c}$, $\Theta = \psi$ integrals of the form $\vec{\mu} = \text{const}$. Therefore, from (6), for (11) there follow the equations of the zeroth approximations $\mathbf{c}, \vec{\mu}$:

$$\dot{\mathbf{c}} = \frac{1}{T} \int_0^T \left\{ (\varepsilon \mathbf{f}_0 \vec{\nabla}_x) + (\varepsilon \vec{\varphi}_0 \vec{\nabla}_\mu) \right\} \vec{\Phi}\, dt, \qquad \dot{\vec{\mu}} = \overline{(\varepsilon \vec{\varphi}_0)} = \frac{1}{T} \int_0^T (\varepsilon \vec{\varphi}_0)\, dt, \tag{12}$$

where $\vec{\nabla}_\mu = \left\{ \dfrac{\partial}{\partial \mu_1}, \dfrac{\partial}{\partial \mu_2}, \dots, \dfrac{\partial}{\partial \mu_m} \right\}$, $\vec{\varphi}_0 \equiv \vec{\varphi}\big|_{\varepsilon=0}$. The equation $\dot{\vec{\mu}} = \overline{(\varepsilon \vec{\varphi}_0)}$ means that, in the zeroth approximation, the true velocity $\varepsilon\vec{\varphi}$ of the parameter $\vec{\mu}$ is replaced by the velocity $\overline{(\varepsilon \vec{\varphi}_0)}$ averaged over a period. For the special case of system (11) with one degree of freedom, equations (12) were derived in [3].

4. **Oscillatory systems with slowly varying parameters, close to canonical ones.** Suppose the unperturbed system has the form

$$\dot{\mathbf{q}} = \vec{\nabla}_p H(\mathbf{p}, \mathbf{q}, \vec{\mu}), \qquad \dot{\mathbf{p}} = -\vec{\nabla}_q H(\mathbf{p}, \mathbf{q}, \vec{\mu}) \quad (\vec{\mu} = \text{const}), \tag{13}$$

and the perturbed system is written in the form

$$\dot{\mathbf{q}} = \vec{\nabla}_p H(\mathbf{p}, \mathbf{q}, \vec{\mu}) - \varepsilon \mathbf{f}_p(\mathbf{p}, \mathbf{q}, \vec{\mu}, \varepsilon), \qquad \dot{\mathbf{p}} = -\vec{\nabla}_q H(\mathbf{p}, \mathbf{q}, \vec{\mu}) + \varepsilon \mathbf{f}_q(\mathbf{p}, \mathbf{q}, \vec{\mu}, \varepsilon),$$

$$\dot{\vec{\mu}} = \varepsilon \vec{\varphi}(\mathbf{p}, \mathbf{q}, \vec{\mu}, \varepsilon). \tag{14}$$

If the periodic solutions (13) depend on $2n$ arbitrary constants and $\vec{\mu}$, then, applying formulas (12) to (14), we derive the energy equation

$$\dot{E} = \frac{1}{T} \oint (\varepsilon f_{q_0}\, dq + \varepsilon f_{p_0}\, dp) + \frac{1}{T} \int_0^T (\varepsilon \vec{\varphi}_0 \vec{\nabla}_\mu) H\, dt. \tag{15}$$

System (8) and equation (9) are a particular case of (14), (15). The physical meaning of (15): the rate of change of E is equal to the mean power of the perturbing forces εf_q, εf_p over a period, added to the power expended on changing $\vec{\mu}$. For the particular case of system (14) with one degree of freedom, equation (15) was derived in ([3],[4]). Independently of (4), the energy equation for canonical systems with one degree of freedom was obtained in ([5]).

Let us now consider the action integral $I = \oint p\, dq$, taken over the cycles (13). Since I may be regarded as one of the integrals (13), applying formula (12) to (14), after a number of transformations we derive the equation for I:

$$\dot{I} = \oint (\varepsilon f_{q_0}\, dq + \varepsilon f_{p_0}\, dp) + \int_0^T \left[\varepsilon \vec{\varphi}_0 - \overline{(\varepsilon \vec{\varphi}_0)} \right] \vec{\nabla}_\mu H\, dt. \tag{16}$$

The physical meaning of (16): the rate of change of I is equal to the work of the perturbing forces over a period, added to the virtual work of changing $\vec{\mu}$ with velocity $\varepsilon \vec{\varphi}_0 - \overline{(\varepsilon \vec{\varphi}_0)}$, which is the deviation of the velocity of $\vec{\mu}$ from its mean value. For the particular case of system (14) with one degree of freedom, equation (16) was derived in ([3],[4]). Some results of previous works ([6]) may be obtained from (16).

Suppose now that $\vec{\mu}$ changes in (14) uniformly, i.e. $\dot{\vec{\mu}} = \varepsilon \vec{\varphi} \equiv \text{const}$, and that the work $\oint (\varepsilon f_{q_0}\, dq + \varepsilon f_{p_0}\, dp) = 0$ on every cycle (13). Then it follows from (16) that $I = \text{const}$. Integrals that are conserved under perturbations are called adiabatic invariants; one of them for (14), under the indicated conditions, is I. In the physical literature (for example, ([7],[8])) the invariance of I is known when system (14) is canonical, which is a stronger restriction, since in that case $\oint (\varepsilon f_{q_0}\, dq + \varepsilon f_{p_0}\, dp) = 0$ on every closed contour, and not only on the cycles (13).

Remark. $I \equiv \oint p\, dq$ is the Poincaré integral invariant for the unperturbed system (13); but from this, as shown in ([7]), the adiabatic invariance of I for the perturbed system (14) does not follow, and requires an independent proof.

It follows from (16) that in the general case I may also fail to be an adiabatic invariant. In special cases I proves to be an invariant whenever the right-hand side of (16) vanishes.

A number of problems on the asymptotics of solutions of systems similar to (2), (11) are considered in ([9–11]).

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Note: Figure translations are in progress. See original paper for figures.