ON THE COMPLETE CONTINUITY OF THE ADJOINT OPERATOR

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Abstract

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ON THE COMPLETE CONTINUITY OF THE ADJOINT OPERATOR

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Schauder showed ([1]) that the mapping adjoint to a completely continuous linear mapping of a normed space X into a normed space Y is completely continuous. In passing from normed spaces to locally convex spaces, Schauder' s theorem generally loses its force. It turns out, however, that there exist locally convex spaces X (Y) for which Schauder' s theorem is true, whatever Y (X) may be. In the present note two classes of such spaces X, respectively Y, are indicated, and the scope of the second of these classes is investigated*.

1. Let X be a locally convex space over the field K of real or complex numbers. A set $A \subset X$ is called **absolutely convex** if $\lambda x + \mu y \in A$ for any two $x, y \in A$ and any two $\lambda, \mu \in K$ such that $|\lambda| + |\mu| \leq 1$. By ΓM we mean the smallest absolutely convex set containing $M \subset X$.

Definition 1. Let $A, B \subset X$. We shall say that A is **completely bounded with respect to** B, and write $A \prec B$, if for every $\varepsilon > 0$ there exists a finite $F \subset A$ such that $A \subset F + \varepsilon B$.

Lemma 1. If B is absolutely convex, $A \prec B$, and $C \subset A$, then $C \prec B$.

Lemma 2. If $A \prec B$ and, together with each of its points x, A contains the whole segment $[0, x]$ joining 0 to x, then $A \subset L_B$, where L_B is the linear hull of the set B.

This is obtained by applying Lemma 1 to the segments $[0, x]$ $(x \in A)$, taking into account that $L_B = \overline{L_B}$.

Lemma 3. The polar B^0 of a set $B \subset X$ is completely bounded with respect to the polar F^0 of any finite $F \subset L_B$.

Proof. Let $F = \{x_1, \dots, x_n\}$. Consider the mapping $\varphi(f) = (f(x_1), \dots, f(x_n))$ of the adjoint space X' into K^n. There exists $\rho > 0$ such that $F \subset \rho\Gamma B$, and consequently $B^0 = (\Gamma B)^0 \subset \rho F^0$; hence, further, $\varphi(B^0) \subset \rho\varphi(F^0) \subset \rho C$, where

$$C = \{(\xi_1, \dots, \xi_n) \in K^n : \ |\xi_i| \leq 1 \ (i = 1, \dots, n)\},$$

i.e. $\varphi(B^0)$ is bounded in K^n. Since every bounded set in K^n is completely bounded, it follows that, for each $\varepsilon > 0$, there exist $f_1, \dots, f_m \in B^0$ such that

$$\varphi(B^0) \subset \bigcup_{i=1}^{m} [\varphi(f_i) + \varepsilon C].$$

But then

$$B^0 \subset \bigcup_{i=1}^{m} (f_i + \varepsilon F^0).$$

Theorem 1. If $A \prec B$ and, together with each of its points x, A contains the whole segment $[0, x]$, then $B^0 \prec A^0$.

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* For definitions of the general concepts of the theory of locally convex spaces used below, see, for example, abstract No. 2760 (of Dieudonné' s survey article) in *RZhMat.* for 1955.

Proof. Since $A \prec B$, for every $\varepsilon > 0$ there exists a finite $F_1 \subset A$ such that

$$A \subset F_1 + \frac{\varepsilon}{4} B.$$

But

$$F_1 + \frac{\varepsilon}{4} B \subset \Gamma \times \left(2F_1 \cup \frac{\varepsilon}{2} B\right).$$

Passing to polars, we obtain

$$\frac{1}{2} F_1^0 \cap \frac{2}{\varepsilon} B^0 \subset A^0,$$

or

$$F^0 \cap 2B^0 \subset \varepsilon A^0,$$

where $F = \frac{2}{\varepsilon} F_1$. Hence it follows easily that

$$(f + F^0) \cap B^0 \subset f + \varepsilon A^0$$

for all $f \in B^0$. But since, by Lemma 2, $F \subset L_A \subset L_B$, it follows, by Lemma 3, that there exist $f_1, \dots, f_n \in B^0$ such that

$$B^0 \subset \bigcup_{i=1}^{n} (f_i + F^0).$$

Consequently,

$$B^0 \subset \bigcup_{i=1}^{n} (f_i + F^0) \cap B^0 \subset \bigcup_{i=1}^{n} (f_i + \varepsilon A^0).$$

Definition 2. We shall call $A \subset X$ **strongly totally bounded** if there exists a bounded $B \subset X$ such that $A \prec B$.

It is clear that every strongly totally bounded set is totally bounded. The converse is obviously true for normed spaces, but is no longer true, for example, when their topology is replaced by the weak topology. Locally convex spaces in which every totally bounded set is strongly totally bounded will be called **spaces of type** (N).

Locally convex spaces in which every bounded set is totally bounded will be called **spaces of type** (M_0).

2. Let X and Y be locally convex spaces. By X' and Y' we shall mean the conjugate spaces endowed with the strong topology. Let φ be a continuous linear mapping of X into Y, and let φ' be the continuous linear mapping of Y' into X', conjugate to it, defined by the formula

$$\varphi'(g)(x) = g(\varphi(x)) \qquad (x \in X,\ g \in Y').$$

Lemma 4 ([2]). If $\varphi(A) \subset B$, then

$$\varphi'(B^0) \subset A^0.$$

Definition 3. A linear mapping of X into Y is called **totally continuous** (**totally bounded**, **bounded**) if there exists in X a neighborhood of zero whose image in Y has bicompact closure (is totally bounded, is bounded), and **quasi-totally bounded** if the image of every bounded set from X is totally bounded in Y.

Theorem 2. The mapping φ', conjugate to any bounded (and, a fortiori, to any totally continuous) mapping φ of a space X of type (M_0) into any locally convex space Y, is totally continuous.

Proof. Let U be a neighborhood of zero in X whose image $\varphi(U) = B$ is bounded in Y. By Lemma 4, $\varphi'(B^0) \subset U^0$. But since in X every bounded set is totally bounded, U^0 is bicompact in X' (([3]), Theorem 1.8). Consequently, $\varphi'(B^0)$ has bicompact closure $\varphi'(B^0)$ in X', which proves the theorem, since B^0 is a neighborhood of zero in Y'.

Lemma 5. The polar U^0 of a neighborhood of zero U from X is complete in X'.

Proof. U^0 is bicompact and, consequently, complete in X' endowed with the weak topology determined by the space X. But from completeness of a set in the weak topology there follows its completeness also in the strong topology, as follows easily, for example, from Proposition 8 § 1 Ch. I in ([2])*.

Theorem 3. The mapping φ', conjugate to any totally bounded (and, a fortiori, to any totally continuous) mapping φ of any locally convex space X into a space Y of type (N), is totally continuous.

Proof. Let U be a convex neighborhood of zero in X whose image $\varphi(U) = A$ is totally bounded in Y. Since Y is a space–

* The completeness of U^0 in X' can be proved still more simply by proceeding from the definition of completeness by means of fundamental generalized sequences ([4]).

of type (N), then there exists a bounded $B \subset Y$ such that $A \prec B$. By Theorem 1, then $B^0 \prec A^0$, whence it follows that $\varphi'(B^0) \prec \varphi'(A^0)$. But, by Lemma 4, $\varphi'(A^0) \subset U^0$. Consequently, $\varphi'(B^0) \prec U^0$, and since U^0 is bounded in X' ([2], Ch. III, § 3, Proposition 7), it follows that $\varphi'(B^0)$, and with it $\overline{\varphi'(B^0)}$, is completely bounded in X'. On the other hand, by Lemma 2 there exists $\rho > 0$ such that $A \subset \rho\Gamma B$. Hence $\varphi'(B^0) = \varphi'((\Gamma B)^0) \subset \rho U^0$; further, $\overline{\varphi'(B^0)} \subset \overline{\rho U^0} = \rho U^0$, and, since by Lemma 5 $\rho U^0 = (\rho^{-1}U)^0$ is complete, $\overline{\varphi'(B^0)}$ is complete. But then $\varphi'(B^0)$, as a complete completely bounded set, is bicompact, which proves the theorem, since B^0 is a neighborhood of zero in Y'.

Theorems 2 and 3 admit the converse: every space X (Y) possessing the property asserted in Theorem 2 (3) is a space of type (M_0) $((N))$.

Theorems 2 and 3 also follow from the following proposition ([5]):

Theorem 4. *A mapping conjugate to the result of imposing a bounded mapping on a quasi-completely bounded one is completely continuous.*

3. The following propositions show that the class $\mathcal{N}$ of all spaces of type (N) includes, if not all, then the overwhelming majority of functional spaces (in their nonweakened topology) presently used in analysis.

Theorem 5. *Every metrizable locally convex space X is a space of type (N).*

Proof. By the assumption of the theorem, in X there exists a countable fundamental system of absolutely convex neighborhoods of zero $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$. Let $A \subset X$ be completely bounded. Then ΓA is completely bounded ([2], Ch. II, § 4, Proposition 2), so that for every n there exist $x_{n,1}, \dots, x_{n,i_n} \in \Gamma A$ such that

$$\Gamma A \subset \bigcup_{i=1}^{i_n} \left(x_{n,i} + \frac{1}{4n} U_n \right).$$

Put

$$A_{n,i} = (\Gamma A) \cap \left(x_{n,i} + \frac{1}{4n} U_n \right), \qquad B_{n,i} = x_{n,i} + n(A_{n,i} - x_{n,i}),$$

$$B_n = \bigcup_{i=1}^{i_n} B_{n,i}, \qquad B = \Gamma \bigcup_{n=1}^{\infty} B_n.$$

$A \prec B$. Indeed, let $\varepsilon > 0$ and $n > \frac{1}{\varepsilon}$. We have $\varepsilon B \supset \frac{1}{n} B \supset$

$$\supset \frac{1}{n} B_{n,i} \supset A_{n,i} - \frac{n-1}{n} x_{n,i} \qquad (i = 1, \dots, i_n),$$

whence

$$\bigcup_{i=1}^{i_n} A_{n,i} \subset \bigcup_{i=1}^{i_n} \left(\frac{n-1}{n} x_{n,i} + \varepsilon B \right),$$

so that $\Gamma A \prec B$. But then, by Lemma 1, also $A \prec B$.

B is completely bounded. Indeed, since $A_{n,i} - x_{n,i} \subset 2(\Gamma A) \cap \frac{1}{4n} U_n$, we have

$$\bigcup_{m=1}^{n-1} B_m = \bigcup_{m=1}^{n-1} \bigcup_{i=1}^{i_m} B_{m,i} \subset \bigcup_{m=1}^{n-1} \bigcup_{i=1}^{i_m} (x_{m,i} + 2m\Gamma A) \subset (2n-1)\Gamma A$$

and

$$\bigcup_{m=n}^{\infty} B_m = \bigcup_{m=n}^{\infty} \bigcup_{i=1}^{i_m} B_{m,i} \subset \bigcup_{m=n}^{\infty} \bigcup_{i=1}^{i_m} \left(x_{m,i} + \frac{1}{4} U_m \right) \subset \Gamma A + \frac{1}{4} U_n.$$

Thus

$$\bigcup_{m=1}^{\infty} B_m \subset 2n\, \Gamma A + \frac{1}{4} U_n,$$

and since $2n\Gamma A + \frac{1}{4} U_n$ is absolutely

is convex, then $B \subset 2n\Gamma A + \frac{1}{4} U_n$. But $2n\Gamma A$ is totally bounded together with ΓA; therefore there exist $y_{n,1}, \dots, y_{n,j_n} \in 2n\Gamma A$ such that

$$2n\Gamma A \subset \bigcup_{j=1}^{j_n} \left(y_{n,j} + \frac{1}{4} U_n \right),$$

and, consequently,

$$B \subset \bigcup_{j=1}^{j_n} \left(y_{n,j} + \frac{1}{2} U_n \right).$$

Choosing one point in each nonempty intersection $B \cap \left(y_{n,j} + \frac{1}{2}U_n\right)$, for these points $z_{n,1}, \dots, z_{n,k_n} \in B$ we shall have

$$B \subset \bigcup_{k=1}^{k_n} (z_{n,k} + U_n).$$

Thus, for every bounded $A \subset X$ there exists a bounded (indeed, totally bounded) $B \subset X$ such that $A < B$. Hence $X \in \mathcal{N}$.

Theorem 6. *The inductive limit X of a sequence of spaces X_n of type (N), with homeomorphic embeddings $X_n \to X_{n+1}$, is a space of type (N).*

This follows from the fact that, as is easy to see, every bounded set $A \subset X$ is contained and bounded in some X_n.

Theorems 5 and 6 show that $\mathcal{N}$ contains all spaces (LF) and, in particular, the space D of "finite" functions ([6]).

Theorem 7. *If X is a t-space of type $(\bar{S})$ ([5]), then $X' \in \mathcal{N}$.*

This follows directly from the reflexivity of the space X and Theorems 9 and 11 of ([5]), which show that in X' every bounded set is strongly totally bounded.

From Theorem 7 it follows, in particular, that $\mathcal{N}$ contains the space D' of "distributions" ([6]), as well as all spaces (LN^*) ([7]).

Let us note, however, that there exist reflexive (F)-spaces X for which $X' \notin \mathcal{N}$.

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Note: Figure translations are in progress. See original paper for figures.