POSITIVE FUNCTIONALS ON ALGEBRAS

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Abstract

Full Text

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Let A be a commutative algebra over the field of real numbers with identity element e. Denote by $\mathfrak{M}$ the space of all homomorphisms M of the algebra A into the field of real numbers satisfying the condition $e(M) = 1$ (i.e., the maximal ideals). We introduce a topology on the set $\mathfrak{M}$ as the weak topology defined by the family of all functions $x(M)$, $x \in A$. An element $x \in A$ is called non-negative if for every homomorphism $M \in \mathfrak{M}$ the number $x(M) \geq 0$. A linear functional F on the algebra A is called positive if $F \neq 0$ and for every non-negative element $x \in A$ we have $F(x) \geq 0$.

If F is positive, then $F(x^2) \geq 0$ for $x \in A$, since $x^2(M) = (x(M))^2 \geq 0$ for $M \in \mathfrak{M}$. But the converse is false. For example, if A is the algebra of all polynomials $p(t, s)$ in two variables with real coefficients, then $\mathfrak{M}$ will be the plane of the variables (t, s). In this case, as R. B. Zarkhina showed ([1]), on A there exists a linear functional F, not identically zero, satisfying the condition $F(x^2) \geq 0$ for every $x \in A$, but not positive.

In the present note we give a necessary and sufficient condition for the positivity of a functional on an algebra with a countable number of generators.

Theorem. Let F be a linear functional on an algebra with a countable number of generators. Then the following three conditions are equivalent:

1. The functional F is positive.
2. On the set $\mathfrak{M}$ there exists a positive measure $\mu(M)$ such that

$$F(x) = \int_{\mathfrak{M}} x(M)\, d\mu(M).$$

3. For every real positive polynomial $p(t_1, \dots, t_n)$ of degree 4 and any $x_1, \dots, x_n \in A$ the inequality

$$F(p(x_1, \dots, x_n)) \geq 0$$

holds.

We note that in the formulation of condition 3 of this theorem, positive polynomials of degree 4 cannot be replaced by positive polynomials of degree 2.

Proof. It is enough to prove that condition 3 implies condition 2. Let $\{x_\alpha\}$ be a linear basis of the algebra A, where $\alpha \in I$, I being the set of all integers. Put

$x_0 = e$ and $F(e) = 1$. To each $\alpha > 0$, $\alpha \in I$, assign a one-dimensional space E_α. Let

$$E = \prod_{\alpha > 0} E_\alpha$$

be the product of the spaces E_α. We regard $u \in E$ as a linear functional on A, putting $u(x_\alpha) = u_\alpha$, the α-th coordinate of u, and put $u(e) = 1$. Then $\mathfrak{M} \subset E$.

Let $C_4(E)$ be the space of all continuous functions $\varphi(u)$ on E such that $\varphi(u)$ depends only on a finite number of coordinates $u_1, u_2, \dots, u_n$ and there exists a constant c such that

$$|\varphi(u)| \le c \left(\sum_{\alpha=1}^{n} u_\alpha^4 + 1 \right).$$

Let $P_4(E)$ be the subspace of the space $C_4(E)$ consisting of all polynomials $p(u) \in C_4(E)$ in the u_α of degree 4. Define a linear functional $\widetilde{F}$ on $P_4(E)$ by the formula

$$\widetilde{F}(p(u_1, u_2, \dots, u_n)) = F(p(x_1, x_2, \dots, x_n)) \tag{1}$$

for $p(u_1, u_2, \dots, u_n) \in P_4(E)$. From condition 3 it follows that

$$\widetilde{F}(p) \ge 0,$$

when $p(u_1, u_2, \dots, u_n) \ge 0$. From the lemma of M. Krein (^{2}) it follows that $\widetilde{F}$ can be extended to $C_4(E)$ with preservation of positivity, i.e. $\widetilde{F}(\varphi(u)) \ge 0$ for $\varphi(u) \ge 0$ and $\varphi \in C_4(E)$.

Let $C_4'(E)$ be the subspace of the space $C_4(E)$ consisting of all such functions $\varphi(u) \in C_4(E)$ that

$$\lim_{|u_\nu| \to \infty} |\varphi(u)| \Big/ \sum_{\nu=1}^{n} u_\nu^4 = 0$$

for some n. By Riesz's theorem there exist positive measures $\mu(u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_n})$ such that

$$\widetilde{F}(\varphi) = \int \varphi(u)\, d\mu(u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_n}),$$

when $\varphi(u) \in C_4'(E)$ and $\varphi(u)$ depends only on $u_{\alpha_1}, \dots, u_{\alpha_n}$. Note that

$$\int d\mu(u_{\alpha_1}, \dots, u_{\alpha_n}) = F(e) = 1.$$

By A. N. Kolmogorov' s theorem (^{2}), on E there exists a positive measure $\mu(u)$ such that

$$\widetilde{F}(\varphi) = \int_E \varphi(u)\, d\mu(u)$$

for $\varphi \in C_4'(E)$. Therefore, if $\varphi \in C_4(E)$ and $\varphi(u) \geq 0$, then

$$\widetilde{F}(\varphi) \geq \int_E \varphi(u)\, d\mu(u), \tag{2}$$

since $\varphi(u)$ can be approximated by a monotone sequence of positive functions from $C_4'(E)$.

For any $\alpha, \alpha' \in I$ there exist real numbers $a_{\alpha\alpha'}^{(k)}$ such that

$$x_\alpha x_{\alpha'} = \sum_k a_{\alpha\alpha'}^{(k)} x_k,$$

where $a_{\alpha\alpha'}^{(k)} = 0$ for all but a finite number of k.

Let $E_{\alpha\alpha'}$ be the subspace of the space E consisting of those elements $u \in E$ which satisfy the condition

$$L_{\alpha,\alpha'}(u) \equiv u(x_\alpha)u(x_{\alpha'}) - \sum_k a_{\alpha\alpha'}^{(k)} u(x_k) \equiv u_\alpha u_{\alpha'} - \sum_k a_{\alpha\alpha'}^{(k)} u_k.$$

Then

$$\mathfrak{M} = \bigcap_{\alpha,\alpha' \in I} E_{\alpha\alpha'}.$$

But from (2) it follows that

$$0 = F\left(\left(x_\alpha x_{\alpha'} - \sum_k a_{\alpha\alpha'}^{(k)} x_k\right)^2\right) = \widetilde{F}(L_{\alpha\alpha'}(u)^2) \geq \int_E L_{\alpha\alpha'}(u)^2\, d\mu(u).$$

Therefore

$$\int_{E-E_{\alpha\alpha'}} d\mu(u) = 0$$

and, consequently,

$$\int_{E-\mathfrak{M}} d\mu(u) = 0.$$

Thus it has been proved that

$$\widetilde{F}(\varphi(u)) = \int_{\mathfrak{M}} \varphi(M)\, d\mu(M) \tag{3}$$

for $\varphi(u) \in C_4'(E)$. If $x \in A$, then there exist a_α such that $x = \sum_{\alpha=1}^{n} a_\alpha x_\alpha$.

From (1) and (3) we obtain that

$$F(x) = F\left(\sum a_\alpha x_\alpha\right) = \widetilde{F}\left(\sum a_\alpha u_\alpha\right) = \int_{\mathfrak{M}} \left(\sum a_\alpha u_\alpha\right) d\mu(u) =$$

$$= \int_{\mathfrak{M}} \sum a_\alpha u(x_\alpha)\, d\mu(u) = \int_{\mathfrak{M}} u(x)\, d\mu(u).$$

Therefore $F(x)$ satisfies condition (2).

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3. A. N. Kolmogorov, Basic concepts of probability theory, Moscow–Leningrad, 1936.

Note: Figure translations are in progress. See original paper for figures.