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Abstract

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MATHEMATICS

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SOLUTION OF PARABOLIC EQUATIONS OF ARBITRARY ORDER BY THE METHOD OF NETS

(Presented by Academician S. L. Sobolev on 21 XI 1957)

In the literature, the question of solving parabolic equations by the method of nets has been considered only as applied to equations of the second order. In the present note some facts known for the case $m = 1$ are generalized to the equation

$$\frac{\partial U}{\partial t} + (-1)^m \frac{\partial^{2m} U}{\partial x^{2m}} = 0, \qquad m = 2, 3, \dots, \tag{1}$$

Let it be required to find a function U satisfying in the domain $D(0 < x < 1;\ 0 < t \leq T)$ equation (1) and the following boundary conditions:

$$U(x, 0) = f(x) \qquad (0 < x < 1);$$

$$\frac{\partial^{2p} U(0,t)}{\partial x^{2p}} = \frac{\partial^{2p} U(1,t)}{\partial x^{2p}} = 0 \qquad (p = 0, 1, \dots, m-1;\ 0 \leq t \leq T). \tag{2}$$

Let us write the net equation

$$\frac{u_{i,k+1} - u_{i,k}}{l} + a(-1)^m \frac{\Delta^{2m} u_{i-m,k+1}}{h^{2m}} + (1-a)(-1)^m \frac{\Delta^{2m} u_{i-m,k}}{h^{2m}} = 0$$

$$\left(i = 1, 2, \dots, n-1;\ k = 0, 1, \dots, \left[\frac{T}{l}\right] - 1\right), \tag{3}$$

where $u_{ik} = u(ih, kl)$, $h = 1/n$ (h and l are the steps, respectively, along the axes x and t),

$$\Delta^{2m} u_{i-m,k} = \sum_{j=0}^{2m} (-1)^j C_{2m}^j u_{i+m-j,k},$$

$$C_{2m}^j = \frac{(2m)!}{j!(2m-j)!}, \qquad 0 \le a \le 1.$$

Substituting into equation (3) the expansions of a sufficiently smooth function $U(x,t)$ in Taylor series in a neighborhood of the point $x = ih,\ t = (k + \frac{1}{2})l$, and using the relation

$$\sum_{j=0}^{m-1} (-1)^j C_{2m}^j (m-j)^{2\gamma} = \begin{cases} 0, & \text{if } \gamma = 0, 1, \dots, m-1, \\ \dfrac{(2m)!}{2}, & \text{if } \gamma = m, \end{cases}$$

we can formally write

$$\begin{aligned} & U_{i,k+1} - U_{i,k} + a\,\frac{l(-1)^m}{h^{2m}}\,\Delta^{2m} U_{i-m,k+1} + (1-a)\frac{l(-1)^m}{h^{2m}}\,\Delta^{2m} U_{i-m,k} \\ & = 2\left[\frac{l}{2}\frac{\partial}{\partial t} + \left(\frac{l}{2}\right)^3 \frac{1}{3!}\frac{\partial^3}{\partial t^3} + \cdots\right] U_{i,k+1/2} + \frac{2l(-1)^m}{h^{2m}} \sum_{j=0}^{m-1} (-1)^j C_{2m}^j \sum_{\gamma=2m}^{\infty} \frac{1}{\gamma!} \times \\ & \quad \times \left\{ \sum_{\substack{\alpha=1 \\ (\alpha \text{ even})}}^{\gamma} C_\gamma^\alpha \left(\frac{l}{2}\right)^{\gamma-\alpha} (m-j)^\alpha h^\alpha \left[a + (-1)^{\gamma-\alpha}(1-a)\right] \frac{\partial^\gamma U_{i,k+1/2}}{\partial t^{\gamma-\alpha}\partial x^\alpha} \right\} \\ & = l\left[\frac{\partial U_{i,k+1/2}}{\partial t} + (-1)^m \frac{\partial^{2m} U_{i,k+1/2}}{\partial x^{2m}} + \frac{l(-1)^m}{2}(2a-1)\frac{\partial^{2m+1} U_{i,k+1/2}}{\partial t\,\partial x^{2m}} + O^2(l^2+h^2)\right]. \end{aligned}$$

It follows from this that, for any sufficiently smooth function U, the left-hand side of the difference equation (3), for sufficiently small h and l, approximates (in the domain of definition of the function U) the left-hand side of the given differential equation (1) with error $O(l+h^2)$ in the case $a \ne 1/2$, and $O(l^2+h^2)$ in the case $a = 1/2$.

The "initial" condition for equation (3), obviously, has the form

$$u_{i,0} = f(ih) \qquad (i = 1, 2, \dots, n-1). \tag{4}$$

Instead of the left boundary condition $\partial^{2p} U(0,t)/\partial x^{2p} = 0$ $(p = 0, 1, \dots, m-1)$, it is natural to take, with an error not exceeding $O(h^2)$, the expression

$$\sum_{j=0}^{2p}(-1)^j C_{2p}^j u_{p-j,k} = 0 \qquad \left(p = 0, 1, \dots, m-1;\ k = 0, 1, \dots, \left[\frac{T}{l}\right]\right).$$

From this we directly obtain the left "boundary" condition for the difference equation (3):

$$u_{-q,k} = -u_{q,k} \qquad \left(q = 0, 1, \dots, m-1;\ k = 0, 1, \dots, \left[\frac{T}{l}\right]\right). \tag{5}$$

In a completely analogous way one obtains the right "boundary" condition for equation (3):

$$u_{n+q,k} = -u_{n-q,k} \qquad \left(q = 0, 1, \dots, m-1;\ k = 0, 1, \dots, \left[\frac{T}{l}\right]\right). \tag{6}$$

By direct calculation one can verify that the system of equations (3)–(6) can be written in matrix form as follows:

$$Au^{(k+1)} = Bu^{(k)} \qquad \left(k = 0, 1, \dots, \left[\frac{T}{l}\right] - 1\right), \tag{7}$$

where

$$u^{(k)} = \{u_{1,k}, u_{2,k}, \dots, u_{n-1,k}\};$$

$$A = E + a\,\frac{l(-1)^m}{h^{2m}}\,C^m; \qquad B = E + (1-a)\frac{l(-1)^m}{h^{2m}}\,C^m,$$

E is the identity matrix;

$$C = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -2 \end{pmatrix}.$$

Here the eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$ of the matrices A and B, respectively, have the form

$$\lambda_i(A) = 1 + a\frac{l}{h^{2m}}2^{2m}\sin^{2m}\frac{i\pi}{2n}; \qquad \lambda_i(B) = 1 - (1-a)\frac{l}{h^{2m}}2^{2m}\sin^{2m}\frac{i\pi}{2n}$$

$$(i = 1, 2, \dots, n-1).$$

Problem (7), for sufficiently small h and l, is an algebraic approximation to the given differential problem (1), (2).

Theorem. 1) For any $0 \le a \le 1$, $k = 0, 1, \dots, \left[\frac{T}{l}\right] - 1$, the system of linear algebraic equations (7) is solvable.

2) If $1/2 \le a \le 1$, then the difference equation (7) is absolutely, i.e. for arbitrary h and l, stable.

3) If $0 \le a < 1/2$, then a necessary and sufficient condition for the stability of the difference equation (7) is the inequality

$$l \le \frac{h^{2m}}{(1-2a)\,2^{2m-1}}. \tag{8}$$

4) Let the solution U of problem (1), (2) exist and have, in $D(0 < x < 1,\ 0 \le t \le T)$, derivatives $\partial^{2m+2}U/\partial t^2 \partial x^{2m}$, $\partial^{2m+2}U/\partial x^{2m+2}$ bounded in absolute value. Then for all l in the case $1/2 \le a \le 1$, and for all l satisfying condition (8) in the case $0 \le a < 1/2$, as $l \to 0$, $h \to 0$, there is convergence in the mean

$$\left(\text{in the sense } (u^{(k)}, v^{(k)}) = \frac{1}{n-1} \sum_{i=1}^{n-1} u_{ik} v_{ik}\right)$$

with respect to i and uniformly with respect to k, of the solution of problem (7) to the solution of the given problem (1), (2). Moreover, the rate of this convergence is determined by the quantity $O(l + h^2)$ in the case $a \ne 1/2$, and $O(l^2 + h^2)$ in the case $a = 1/2$.

Remark 1. For $m = 1$, problem (1), (2) becomes the simplest, well-studied parabolic problem; in this case the stability condition (8) becomes the corresponding known stability condition (for example, for $a = 0$, $m = 1$, from (8) we have $l \le h^2/2$).

Remark 2. For $a = 0$, problem (7) can be solved explicitly. To this end, setting $u_{ik} = X_{iT}k$, we separate variables in equation (3) for $a = 0$:

$$\frac{T_{k+1}}{T_k} = \frac{X_i + \dfrac{l(-1)^{m+1}}{h^{2m}} \Delta^{2m} X_{i-m}}{X_i} = \mu.$$

Hence we immediately obtain

$$T_k = \mu_0^{kT} \qquad \left(k = 0, 1, \dots, \left[\frac{T}{l}\right]\right);$$

$$X_i + \frac{l(-1)^{m+1}}{h^{2m}} \Delta^{2m} X_{i-m} = \mu X_i \qquad (i = 1, 2, \dots, n-1), \tag{9}$$

$$X_{-q} = -X_q, \qquad X_{n+q} = -X_{n-q} \qquad (q = 0, 1, \dots, m-1).$$

Using the relation (see ([1]), p. 39)

$$\sin^{2r} y = \frac{1}{2^{2r}} \left\{ 2 \sum_{k=0}^{r-1} (-1)^{r-k} C_{2r}^k \cos 2(r-k)y + C_{2r}^r \right\},$$

it is easy to verify that the solution of the eigenvalue problem (9) has the form

$$\mu_p = 1 - l \left(\frac{2 \sin(p\pi h/2)}{h} \right)^{2m};$$

$$X_i^{(p)} = \sin p\pi i h \quad (p = 1, 2, \dots, n-1; \; i = 1, 2, \dots, n-1),$$

and, consequently, the desired solution of problem (7) can be written in the form

$$u_{ik} = \sum_{p=1}^{n-1} a_p \mu_p^k \sin p\pi i h \left(a_p = \frac{2}{n-1} \sum_{q=1}^{n-1} f(qh) \sin p\pi q h \right). \tag{10}$$

Comparing the expression

$$U_{ik} = \sum_{p=1}^{\infty} c_p e^{-(p\pi)^{2m} kl} \sin p\pi i h \left(c_p = 2 \int_0^1 f(\zeta) \sin p\pi\zeta \, d\zeta \right),$$

which determines the solution of problem (1), (2), with expression (10), one can, under the condition $l \leq h^{2m}/2^{2m-1}$, prove the uniform convergence of u_{ik} to U_{ik} as $h \to 0$, under quite weak assumptions on the initial function $f(x)$ (f is continuous except at a finite number of points at which it may have a finite jump, and has bounded variation on $(0, 1)$).

Remark 3. For equation (1), by linking in one mesh equation the same number of nodes as in (3), for $\alpha \neq 0$ and $\alpha \neq 1$, one can write more accurate mesh approximations. For example, for $\alpha = 1/6$ the mesh equation

$$(1 + \alpha\Delta^2) \frac{u_{i,k+1} - u_{i,k}}{l} + \frac{1}{2} \left(\frac{\Delta^4 u_{i-2,k+1}}{h^4} + \frac{\Delta^4 u_{i-2,k}}{h^4} \right) = 0,$$

from which, for $\alpha = 0$, equation (3) is obtained for $m = 2$, $a = 1/2$, approximates equation (1) ($m = 2$) with error $O(l^2 + h^4)$.

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CITED LITERATURE

1. I. M. Ryzhik, I. S. Gradshteyn, *Tables of Integrals, Sums, Series, and Products*, 1951.

Note: Figure translations are in progress. See original paper for figures.