A LINEARIZED ESTIMATE OF THE ERROR OF NUMERICAL INTEGRATION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE BASIC PROBLEM OF EXTERIOR BALLISTICS

Let the solution of the problem

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. N. KOTOVA

A LINEARIZED ESTIMATE OF THE ERROR OF NUMERICAL INTEGRATION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE BASIC PROBLEM OF EXTERIOR BALLISTICS

(Presented by Academician V. I. Smirnov, 10 III 1958)

We introduce the necessary notation and definitions. If φ is a vector with components $\varphi^1, \varphi^2, \dots, \varphi^n$, then $\|\varphi\| = \max_{1\le i\le n} \|\varphi^i\|$. If $A = (a_{ik})_1^n$ is a square matrix of order n, then $\{A\}_{ik} = a_{ik}$, $\|A\| = \max_{1\le i\le n} \sum_{k=1}^n |a_{ik}|$. We denote $d\varphi/dt = \dot\varphi$. Let a system of differential equations $\dot\varphi = A\varphi$ be given, where $A = (a_{ik}(t))_1^n$, with $a_{ik}(t)$ continuous functions for $t_0 \le t \le T$. The matrix $X(t)$, whose columns are n linearly independent solutions of the system $\dot\varphi = A\varphi$, with $X(t_0) = E$, is called the matricant of the matrix A and is denoted by the symbol $\Omega_{t_0}^t$ ([1]). In this work we use a linearized estimate of the error of numerical integration obtained by S. M. Lozinskii*. For brevity we shall give the estimate for Adams' extrapolation method, although it is also applicable to his interpolation method.

Let the solution of the problem

$$\dot\varphi = f(t,\varphi), \qquad \varphi(t_0) \text{ given}, \tag{1}$$

be found by a numerical integration method on the interval $[0,T]$, and let Adams' extrapolation formula of order r be applied (we shall say that the method A_r is applied):

$$\varphi_{k+1} = \varphi_k + \left[h(f_k + \alpha_1 \Delta f_{k-1} + \alpha_2 \Delta^2 f_{k-2} + \dots + \alpha_r \Delta^r f_{k-r})\right]_{\text{okr}},$$

where $k = r,\ r+1, \dots, N-1$; $f_k = f(t_k, \varphi_k)$; $t_\nu = t_0 + \nu h$; ν is an integer; $t_0 + Nh \le T$; h is the step of the method ([2]). The vectors φ_k for $k = 0, 1, 2, \dots, r$ must be given. The symbol $[h(\dots)]_{\text{okr}}$ denotes the vector obtained as a result of rounding each component of the vector $h(\dots)$. Let the vector φ_k be the result of numerical integration of (1) at $t = t_k$, and let $\varphi(kh)$ be the value of the solution of (1) at $t = t_k$, and let $\Delta(t_k) = \varphi_k - \varphi(t_k)$. Then the linearized error theory of S. M. Lozinskii gives the following estimate for $\|\Delta(t_k)\|$:

$$\|\Delta(t_k)\| \leq \left\|\Omega_{t_0}^{t_k} J\right\| \varepsilon_0 + \alpha_{r+1} h^{r+1} \int_{t_0}^{t_k} \left\|\Omega_{\xi}^{t_k} J\right\| \left\|\varphi^{(r+2)}(\xi)\right\| d\xi + \frac{\rho}{h} \int_{t_0}^{t_k} \left\|\Omega_{\xi}^{t_k} J\right\| d\xi, \tag{2}$$

where ε_0 is a number such that $\|\varphi_k - \varphi(t_k)\| \leq \varepsilon_0$ for $k = 0, 1, \ldots, r$; ρ is a number such that $\|h(\ldots) - [h(\ldots)]_{\text{okr}}\| \leq \rho$; the matrix $J = J(t)$ is the Jacobi matrix

* Not published. We present this estimate with the permission of S. M. Lozinskii.

for $f(t, \varphi)$ on the solution (1), i.e. $J(t)$ is determined by the formulas

$$\{J(t)\}_{\mu\nu} = \frac{\partial f^{\mu}[t, \varphi(t)]}{\partial \varphi^{\nu}},$$

where $\varphi(t)$ is the exact solution of problem (1).

In exterior ballistics one considers a system of differential equations characterizing the motion of the center of mass of an artillery projectile under the action of gravity and air resistance. Integration of this system is called the principal problem of exterior ballistics (see (3), p. 8). For integration of the system in finite form no method has yet been found; therefore its numerical integration is carried out. To estimate the error of the numerical integration of this system we shall apply formula (2). If time t is taken as the argument, the system has the form (see (3), p. 90):

$$\dot{u} = -cG(v)H(y)u; \quad \dot{w} = -cG(v)H(y)w - g; \quad \dot{x} = u; \quad \dot{y} = w, \tag{3}$$

where $v = \sqrt{u^2 + w^2}$; $c = \text{const} > 0$; $H(y)$ and $G(v)$ are given functions, with x, y measured in meters, and u, w in m/sec.

For $H(y)$ there are tables and many empirical formulas; in the present work the exponential Siacci formula is used, $H(y) = e^{-\alpha y}$, where $\alpha = 1.059 \cdot 10^{-4}$ (3). The function $G(v) = F(v)/v$, where $F(v)$ is called the air-resistance function.

In computing trajectories, various laws of air resistance are used. Mainly the "Siacci law" and the "1943 law" are used; for the Siacci law there are tables and an analytic expression, though a very cumbersome one (4); for the 1943 law there is an analytic expression only for $v \leq 256$ and $v \geq 1410$, i.e., in essence, the 1943 law is given in tabular form. Thus the right-hand sides of (3) are given tabularly, and therefore one cannot speak of a rigorous estimate of the error of numerical integration of (3) (since, in essence, there is no system of differential equations itself), and the linearized error estimate (2) is used. In exterior ballistics one introduces the trajectory element θ, connected with u, w by the formulas $u = v\cos\theta$, $w = v\sin\theta$, where $-\pi/2 < \theta_c \leq \theta \leq \theta_0 < \pi/2$, with θ_0 the angle of projection of the projectile and θ_c the angle of fall. We shall agree to understand by the symbols u, w, x, y, v, θ simply the set of coordinates, and by the symbols $u(t), w(t), x(t), y(t), v(t), \theta(t)$ the set of the same coordinates on the solution. Let $J = J(t)$ be the Jacobi matrix for the right-hand sides of system (3) on the solution of system (3) satisfying the initial conditions

$$u(0) = u_0, \quad w(0) = w_0, \quad x(0) = 0, \quad y(0) = 0; \tag{4}$$

then the following theorem is valid.

Theorem 1. Let:

1) $\varphi(t)$ be a solution of system (3), satisfying conditions (4) and existing on $[0, T]$;

2) $G(v)$ be a positive, increasing function having continuous derivatives up to order $(r+1)$ inclusive;

3) $H(y) = e^{-\alpha y}, \quad \alpha = \text{const} > 0$;

4) $L(\tau) = \sqrt{2}(1 + T - \tau)$ for $0 \le \tau \le T$;

5) the numbers $T_1 \ge 0$ and $T_2 \le T$ are determined as follows: $T_1 = 0$ if $\theta_0 \le \pi/4$; $T_1 = T$ if $\theta(T) \ge \pi/4$; T_1 is determined from the condition $\theta(T_1) = \pi/4$, if $\theta_0 > \pi/4$ and $\theta(T) < \pi/4$; $T_2 = T$, if $\theta(T) \ge -\pi/4$; T_2 is determined from the condition $\theta(T_2) = -\pi/4$, if $\theta(T) < -\pi/4$;

6) $\Phi_1(\xi) = \alpha\sqrt{2}\{(1 + T - \xi)w(\xi) - (1 + T - T_1)w(T_1) - y(T_1) + y(\xi) -$

$$-g(T_1 - \xi)\left(1 + T - \frac{T_1 + \xi}{2}\right)\} \quad \text{for } 0 \le \xi \le T_1;$$

$$\Phi_2(\xi) = \alpha\sqrt{2}\{(1 + T - \xi)\times$$
$$\times u(\xi) - (1 + T - T_2)u(T_2) - x(T_2) + x(\xi)\} \quad \text{for } T_1 \le \xi \le T_2;$$

$$\Phi_3(\xi) = \alpha\sqrt{2}\{w(T) - (1 + T - \xi)w(\xi) + y(T) - y(\xi) + \tfrac{1}{2}g[(1 + T - \xi)^2 - 1]\}$$
$$\text{for } T_2 \le \xi \le T.$$

Then

$$\left\| \overset{T}{\underset{0}{\Omega}} J \right\| \leqslant L(0)\exp[\Phi_1(0) + \Phi_2(T_1) + \Phi_3(T_2)], \tag{5}$$

$$\int_0^T \left\| \overset{T}{\underset{\xi}{\Omega}} J \right\| \left\|\varphi^{(r+2)}(\xi)\right\| d\xi \leqslant$$

$$\leqslant \exp[\Phi_2(T_1) + \Phi_3(T_2)] \int_0^{T_1} L(\xi)\exp[\Phi_1(\xi)] \left\|\varphi^{(r+2)}(\xi)\right\| d\xi +$$

$$+ \exp[\Phi_3(T_2)] \int_{T_1}^{T_2} L(\xi)\exp[\Phi_2(\xi)] \left\|\varphi^{(r+2)}(\xi)\right\| d\xi +$$

$$+\int_{T_2}^{T} L(\xi)\exp[\Phi_3(\xi)]\left\|\varphi^{(r+2)}(\xi)\right\| d\xi. \tag{6}$$

Remark 1. It can be shown that formula (6) remains valid after replacing $\left\|\varphi^{(r+2)}(\xi)\right\|$ by unity. The result of such a replacement gives an estimate for

$$\int_0^T \left\| \underset{\xi}{\overset{T}{\Omega}} J \right\| d\xi.$$

Remark 2. For any analytic law of air resistance $G(v)$ satisfying condition 2), Theorem 1, together with formula (2), gives for this resistance law a linearized estimate of the error of numerical integration of (3) by the method A_r with step h. Since, however, analytic expressions for the functions $u(t), w(t), x(t), y(t)$ are unknown to us, in the situation under consideration we shall apply Theorem 1 using approximate expressions for these functions obtained from the numerical integration itself of system (3) on the interval $[0, T]$, i.e., we shall apply Theorem 1 for an a posteriori estimate of the error of numerical integration.

Remark 3. It can be shown that the functions $\Phi_i(\xi)$ are decreasing functions; therefore the integrals entering (6) may be estimated without using quadrature formulas, for example in the following way:

$$\int_0^{T_1} L(\xi)e^{\Phi_1(\xi)}\left\|\varphi^{(r+2)}(\xi)\right\| d\xi \leqslant \sum_{i=0}^{n-1} \max_{t_i\leqslant t\leqslant t_{i+1}} \left\|\varphi^{(r+2)}(t)\right\| \exp[\Phi_1(t_i)] \int_{t_i}^{t_{i+1}} L(\xi)\, d\xi,$$

where $0 = t_0 < t_1 < \cdots < t_n = T_1$.

Remark 4. Putting in formula (2) $t_0 = 0,\ t_k = T$, we obtain a formula for estimating $\|\Delta(T)\|$.

The 1943 law of air resistance (we shall denote it by $G_{43}(v)$) is given in tabular form with step $\Delta v = 1$ [4]. Using four points taken from the ballistic tables for $F_{43}(v)$, we obtain

$$G(v) = a_3 v^2 - a_2 v + a_1 - \frac{a_0}{v}, \tag{7}$$

where $a_3 = 6.396 \cdot 10^{-8},\ a_2 = 0.6325 \cdot 10^{-4},\ a_1 = 0.1548,\ a_0 = 26.63$.

The results of computations show that $G(v) = G_{43}(v)$ with accuracy up to 0.6% for $400 \leqslant v \leqslant 1480$, and for $670 \leqslant v \leqslant 1380$ we have $G(v) = G_{43}(v)$ with accuracy up to one unit of the fourth (and last) tabular digit of $G_{43}(v)$.

With the aid of formula (7), the following has been done:

1) For the 1943 law, an improved estimate was obtained for $\left\|\overset{T}{\underset{0}{\Omega}} J\right\|$ and

$$\int_0^T \left\|\overset{T}{\underset{\xi}{\Omega}} J\right\| \|\varphi^{(r+2)}(\xi)\| \, d\xi.$$

These estimates can be formally obtained from the theo-

Theorem 1 by replacing $L(\tau)$ by $1 + T - \tau$ and discarding the factor $\sqrt{2}$ everywhere it occurs.

2) Estimates have been obtained for $\|\varphi^{(r+2)}(t)\|$.

3) The error of the initial ordinates, computed—as is done in exterior ballistics—by the method of successive approximations (see [3], p. 170), has been determined by comparing them with the ordinates computed by means of a Taylor series. Let us note that Theorem 1 is valid for any ballistic law $G(v)$ and for any initial conditions, whereas the improved estimate has been obtained for the 1943 law and for those solutions of system (3) on which $360 \leqslant v(t) < \infty$. We give the results of the examples considered. In solving the examples, formulas (2), (5), (6), and the estimate $\|\varphi^{(r+2)}(t)\|$ obtained with the aid of formula (7) were used. Let $\theta_0 = 50°$, $v_0 = 1000$ m/sec, $c = 0.2$. Denote by T_{v} the instant at which the projectile rises to its greatest height, and by T the instant at which it falls to the ground, and consider the solution on the interval $[0, T]$. If numerical integration is carried out by the method A_3 with $\varepsilon_0 = 0.01$, $h = 0.5$, $\rho = 5 \cdot 10^{-4}$, then $\|\Delta(T)\| \leqslant 90$, $\|\Delta(T_{\text{v}})\| \leqslant 9$. Since $x(T) \doteq 59000$ m, $x(T_{\text{v}}) \doteq 30000$ m, $y(T_{\text{v}}) \doteq 20000$ m, while $B_{\text{d}}(T) \doteq 200$ m, $B_{\text{d}}(T_{\text{v}}) \doteq 100$ m, $B_{\text{v}}(T_{\text{v}}) \doteq 80$ m, where $B_{\text{d}}(t)$, $B_{\text{v}}(t)$ are the median deviations, respectively, in range and height at the instant t [5], the estimate may be regarded as satisfactory. If, however, the numerical integration is carried out by the method A_4 with $\varepsilon_0 = 0.001$, $h = 0.5$, $\rho = 10^{-4}$, then $\|\Delta(T)\| \leqslant 15$, $\|\Delta(T_{\text{v}})\| \leqslant 1.5$.

I express my gratitude to S. M. Lozinskii for his assistance.

Received
20 II 1958

References

[1] F. R. Gantmakher, *Theory of Matrices*, Moscow, 1954, p. 380.
[2] A. N. Krylov, *Lectures on Approximate Calculations*, Moscow–Leningrad, 1950, p. 302.
[3] D. A. Venttsel, Ya. M. Shapiro, *Exterior Ballistics*, Part 1, Moscow–Leningrad, 1939.
[4] Ya. M. Shapiro, *Exterior Ballistics*, Moscow, 1946, pp. 46-48, 310-316.
[5] G. I. Blinov, *Theory of Firing of Ground Artillery*, Part 1, Moscow, 1948, p. 14.